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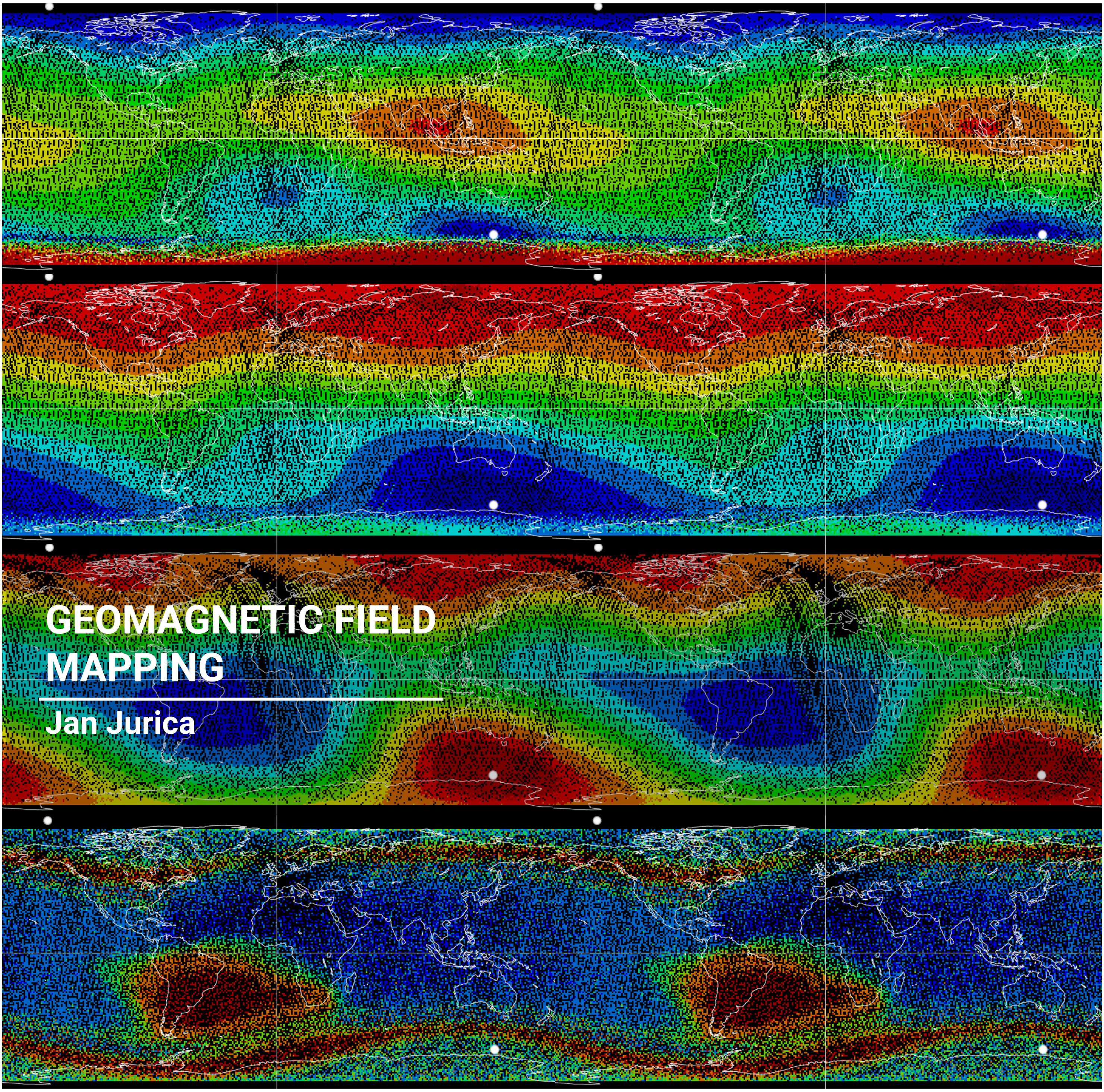

GEOMAGNETIC FIELD MAPPING
Jan Jurica

FROM THE EDITOR

On behalf of the editorial board of the Alumni scientiae Bohemicae Society (ASB), I am delighted to announce the publication of the inaugural issue of the *Journal of the ASB Society* (JASB).
This first issue is an evidence of a hard work over last many months — the first idea emerged at the beginning of the year 2020. It was recognised by the soon-to-be editors of our journal that there is a major gap in the academic publishing landscape; there is not a single journal dedicated to pre-university research. Yet, there is much research done by students at this early stage of their career. There are even competitions encouraging pre-university students to conduct research, e.g. the Students professional activities held annually under the auspices of the Ministry of Education, Youth and Sports of the Czech Republic. JASB aims to provide a platform for pre-university research in all fields of research, including but not restricted to science, mathematics, engineering, humanities and social sciences.

Throughout last year, we approached selected young researchers with requests to contribute. Thanks to an overwhelming interest on their side, we were able to curate five manuscripts for publication in the first issue of our journal. Thus the Journal of the ASB Society was born.

The committee decided that the journal should be published as a multidisciplinary open-access peer-reviewed online journal with at least one issue annually to be printed. This inaugural issue presents articles from the fields of chemistry, mathematics, physics and didactics. We are also pleased to announce that the first issue is dedicated to the first Czech Nobel laureate — Prof. Jaroslav Heyrovský (12th December 1890 – 27th March 1967) — celebrating the 130th anniversary of his birth. This December, it is the 71st anniversary of his Nobel Prize in Chemistry for the development of polarography (1959). By coincidence, also this December, the ASB Society has reached its 5th anniversary. This inaugural issue would absolutely not be possible without the hard work of many people. Thanks are due first to the JASB editorial board, where the idea originated. I am also grateful to the other members of ASB Society and collaborators that drafted the blueprint of our journal as well as to our reviewers, who guarded the academic rigour of the articles. My greatest thanks, however, go to my co-editors Alexandr Zaykov (Managing Editor) and Adam Přáda (Technical Editor), who have so generously given much time, work and their expertise to make this project happen.

We hope that this new platform for young researchers will provide motivation and inspiration, and encourage them to pursue their interests and maybe even to follow Jaroslav Heyrovský in his footsteps.

Jan Hrabovsky
Editor-in-Chief
Faculty of Mathematics and Physics
Charles University

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ABOUT *JOURNAL OF THE ASB SOCIETY* (JASB)

The *Journal of the ASB Society* (JASB) is a multidisciplinary open-access peer-reviewed science journal for young researchers. In the form of full-length articles, rapid communications and special issue articles, JASB aims to promote original high-quality research mainly by undergraduate university and high school students.

The Journal of the Alumni Scientiae Bohemicae Society (JASB) is a pioneering publication set up to provide exclusively undergraduate, young professionals and high school students the opportunity to gain experience in scientific writing. It not only aims to alleviate the first contact hurdle with publishing pressure, but also to encourage and most importantly promote outstanding research by younger scientists. Another opportunity for submitters is to encounter proper writing form early on. All of this effectively leads to the simplification of possible future endeavours in impacted journals.

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Consider submitting your next research paper to the journal.

ABOUT ALUMNI SCIENTIAE BOHEMICAE, Z.S.

ALUMNI SCIENTIAE BOHEMICAE

ASB society is a platform connecting students, young professionals, alumni within student organisations, companies and other institutions not only from the Czech Republic. The main aim is to provide support to young professionals and students from various research and education areas. It is worth mentioning that ASB organises several other activities, such as Chemistry Race (chemistry-based team competition), educational seminars for young professionals and annual training workshops which are focused on the preparation of young professional and scientist for international competitions, such as European Union Contest for Young Scientists (EUCYS), Regeneron International Science and Engineering fair (ISEF), etc. More information can be found on webpage *czechscience.cz* (CZE) or *asbsociety.com* (ENG).

Jaroslav Heyrovský: The story of a mercury drop

Stejskalová, Květoslava, [1], Heyrovský, Michael [1]

The travelling exhibition called "The story of a mercury drop" introduces Jaroslav Heyrovský to the young generations since the year 2009.

Since 2009, which was the 50th anniversary of the award of Nobel prize to Jaroslav Heyrovský on 10.12.1959, the wider public is reminded of this scientist by a travelling exhibition which presents the life and scientific work of Jaroslav Heyrovský not only to those who remember polarography, but also to those interested in natural sciences born later, i.e. to students and pupils.

The purpose of the exhibition is to introduce the personality of Jaroslav Heyrovský not only as a scientist but as a human. The exhibition has been composed of documents which had been deposited in the institute archive for many years, a lot information was drawn from the books about Jaroslav Heyrovský written by his students, or from narrations of his children, pupils and coworkers. A large variety of polarographs from 1924 until the 1990s are exhibited, as well as photographs, written documents, books, publications and film material. For the selection of the exhibits the team examined almost 10 kg of written materials, 200 photographs, 150 slides and 6 km of celluloid films from the 1950s and 60s. Along with some 10 instruments it seemed that there would be sufficient material for creating an exhibition. First preparations started at the beginning of 2008 and the exhibition had its small-scale showing in November 2008 at the the "Week of Science and Technology" organized by the Academy of Sciences of the Czech Republic in its central building at Národní třída. It created a considerable interest from the visitors and the team of authors (three scientists from the J.Heyrovský Institute of Physical Chemistry of the ASCR: K. Stejskalová, M.Heyrovský [died 12.4.2017] and R.Kalvoda [died 2.8.2011]) decided to continue working on it until the end of 2009 and offer a more comprehensive exhibition to the general audience across several places of the Czech Republic.

The exposition is composed of a set of twelve posters (size A0) which, using photographs and documents, introduce the visitors to the life and the scientific work of Jaroslav Heyrovský. Heyrovský's family also lent photographs from the family album to make copies for the exhibition, some of which have not been publicly presented before. Many photographs, slides and written material from the institute´s archives illustrate the research work of J.Heyrovský and his team. The second part of the exhibition consists of instruments (development series of 8-10 polarographs), glass polarographic cells, slides which polarographers lectured with, a display of books on polarography in various world languages. The films made in the 1950s and 60s documenting Heyrovský´s research are projected as a part of the exhibition in an endless loop. The exhibition is completed with an accompanying programme of a number of popularising lectures not only about Jaroslav Heyrovský and his research in polarography, but also about contemporary science and research in the field of physical chemistry in which the scientists at the Jaroslav Heyrovský Institute of Physical Chemistry are engaged today.

The travelling exhibition "THE STORY OF A MERCURY DROP" which has since 2009 been exhibited at 31 places, can also be explored on the web page www.heyrovsky.cz

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Květa Stejskalová *studied chemical engineering at the Technical University in Prague and defended her dissertation work in the subject physical chemistry - heterogeneous catalysis (1995). Since 1989 she worked in the Jaroslav Heyrovský Institute of Physical Chemistry on the fundamental and applied research of catalysis and, ultimately, of electrochemistry. Currently, as the secretary of vice-director for education, she prepares and realizes educational and popularization programs for the institute, which are seen by up to 8000 participants every year. For her work, she was appraised by the V.Náprstek honorary medal for the popularization of science in 2011.*

Michael Heyrovský *(1932-2017) studied chemistry at the Faculty of Science, Charles University (1951-1957). From 1957, he was employed at the Institute of Physical Chemistry, Academy of Sciences of the Czech Republic. Between 1962 and 1965 he did a research study at the Department of Physical Chemistry, University of Cambridge and obtained a PhD with his thesis on "The Electrochemical Photoeffect". He spent two years of research stay (1967 and 1968) at the University of Bamberg, Germany, as an Alexander-von-Humboldt scholar. He is also an author and co-author of over 100 publications on polarography and voltammetry and their applications.*

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Jaroslav Heyrovský: The story of a mercury drop

Stejskalová, Květoslava,[1]* **Heyrovský, Michael**[1]

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"Professor Heyrovský. You are the originator of one of the most important methods of contemporary chemical analysis. Your instrument is extremely simple — just falling droplets of mercury — but you and your collaborators have shown that it can be used for the most diverse purposes ... May I ask you to advance and receive the Nobel Prize for Chemistry for this year from the hands of our King", said professor Arne Ölander, member of the Nobel Committee for Chemistry, in Czech language to the laureate Jaroslav Heyrovský at the award ceremony in the Stockholm Concert Hall on 10th December 1959. The year of 2020 is the year of 130th anniversary of the birth of prof. Jaroslav Heyrovský, the first Czech recipient of the Nobel Prize in Chemistry and 61 years since he received his prize in Stockholm (Sweden).

Key words: Nobel Prize, chemistry, physics, Heyrovský.

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This article was partly inspired and related with the publication presented in the Interface journal of The Electrochemical Society, Vol. 24, No.4, Winter 2015, 36-39.

The young Scientist

Jaroslav Heyrovský was born on 20th December 1890 in Křižovnická street No. 14 in the Old Town district of Prague as the fourth child of Leopold Heyrovský, a professor of Roman Law at the Charles University of Prague, and his wife Klára Hanel. He grew up with his younger brother Leopold (Leo) and three older sisters Klára, Marie and Helena. Together with Leo, they used to collect various fossils on family trips. As a young boy, Jaroslav already started writing a science textbook with his own illustrations, kept small animals at home and helped his science teacher, František Bayer, organise school collections. His secondary school professor Jaroslav Jeništa noticed a sincere interest his young student had in chemistry and physics. One time, young brothers Jaro (Jaroslav) and Leo sent their friends X-ray photos of their hands or aquarium fish as New Year's greetings. Once, they contaminated the whole street with the smoke of salmiac (ammonium chloride) produced by mixing hydrochloric acid with ammonia. From the hobby of his high school, Jaroslav has developed a serious interest in chemistry and physics which he decided to study further. He completed his secondary education at the Academic Grammar School on the Smetana Embankment in Prague. Among his professors was the writer Zikmund Winter; his class-mates were the future writer Karel Čapek or Zdeněk Myslbek, son of the sculptor Václav Myslbek. After his maturita examination, Heyrovský matriculated at the Philosophical Faculty of the Charles University in Prague into physics, chemistry and mathematics courses (an independent Faculty of Science did not yet exist). After a year, when he realized that he could not obtain the necessary scientific education in the field of physical chemistry in Prague, he continued in the study of this subject at the University College London under Sir William Ramsay, where he obtained the Bachelor of Science (BSc) degree in 1913. He continued his scientific work in the electrochemical laboratory of professor F. G. Donnan studying aluminium electrodes. This was unfortunately interrupted by the outbreak of the First World War in 1914.

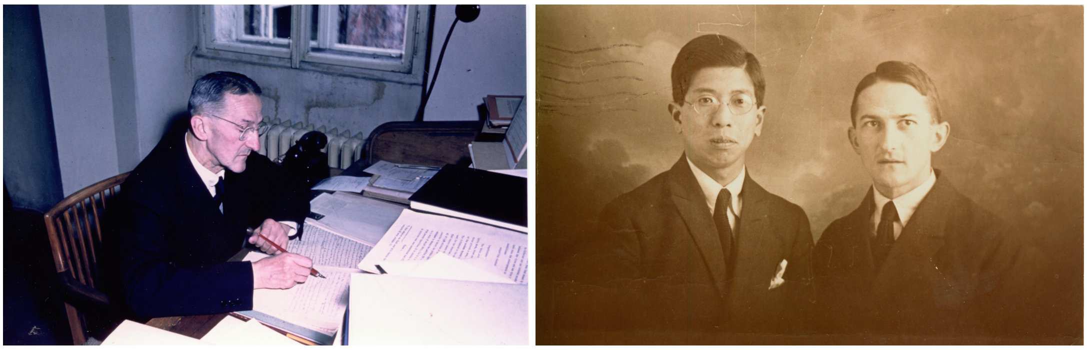

Figure 1: (a) Jaroslav Heyrovský (20/12/1890–27/3/1967) (b) J. Heyrovský with his Japanese coworker M. Shikata (London 1923)

Profesor Kučera and Heyrovský's crucial encounter with the drop

On 27th June 1918 Heyrovský arrived at his rigorous examination, which preceded the defense of his dissertation work, in the uniform of a sanitary corporal. His examiners were professor of inorganic chemistry Bohuslav Brauner (1855–1935), pharmaceutical chemistry professor J.S. Štěrba–Boehm (1878–1939) and professor of experimental physics Bohumil Kučera (1874–1921). Heyrovský's dissertation work "On the electroaffinity of aluminium" was of an exceptional level; the candidate was well known to the examiners, resulting in something resembling more a scientific discussion than an examination. In his dissertation, Heyrovský had a section on aluminium amalgam dropping from a glass capillary into a solution. Although those experiments were not successful, they inspired Kučera to ask a general question about electrocapillarity. Heyrovský answered that question very well and Kučera pointed out the disagreement between the surface tension values measured by his method and by the method suggested by the French physicist G. Lippmann fifty years earlier. To that, the second examiner Brauner, puffing from a thick cigar, added: "That can be solved only by a physical chemist!". This statement was an appeal to Heyrovský, who was then the only specialist in physical chemistry at the whole university. The result of the examination was excellent, resulting in Jaroslav Heyrovský being promoted to a Doctor of Philosophy on 26th September 1918. However, Kučera's problem attracted Heyrovský's attention — was it a new mysterious effect which even a respected scientist, such as Kučera, could not explain? He therefore accepted Kučera's invitation to the university's Physics Institute where Kučera introduced him to the problem in more detail. Heyrovský then started working in Kučera's laboratory in his free time to explain these "anomalies" on electrocapillary curves — the dependency of surface tension on the potential of a dropping electrode. The measurement consisted in weighing drops of mercury which were dropping out of a glass capillary into a solution. The dropping mercury was connected to a source of DC voltage which served as one electrode; the other electrode was the mercury accumulating at the bottom of the cell.

Long and winding are the ways to discoveries

Heyrovský's notebooks from 1921 indicate that only the measurements of electrocapillary curves (i.e. weighing of mercury drops) in various electrolytes did not lead to the discovery. In December 1921, he returned to the study of aluminum complexes by the means of equilibrium electrode potentials, but on 29th December he was trying to measure electrocapillary curves in solutions of aluminum chloride again. In the next few days, even on the New Year's Eve and on the New Year's Day of 1922, he continued measuring the electrocapillary curves in solutions of $MgCl_2$, $BaCl_2$, KCl, LiCl, NaCl, NH_4Cl and $CaCl_2$. It was on New Year's Day that he tried to measure the electric current passing through the dropping and the reference electrode for the first time. Perhaps, he used a galvanometer of a low sensitivity and the resulting curve was not satisfactory. From his notes in the laboratory notebook it appears that he had an new experiment in mind. He borrowed a more sensitive galvanometer from physics professor Záviška. On 9th February 1922, he measured the electrocapillary curves in a solution of NaCl again and observed something unusual which he commented: "Something is happening on the top (of the curve), but, at the moment, there is no time to investigate what that is." At last, on 10th February 1922, he included a mirror galvanometer in the measuring circuit and electrolysed 1 mol dm^{-3} solution of sodium hydroxide. Already with the slightest voltage applied, a small deflection appeared

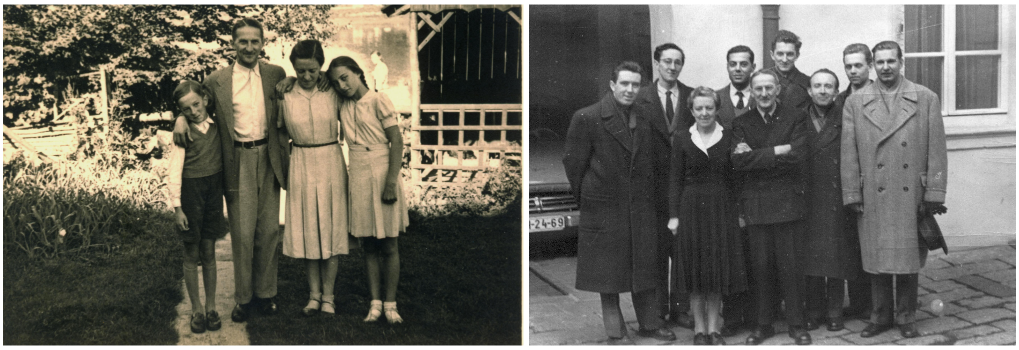

Figure 2: (a) Jaroslav Heyrovský with his wife Marie, son Michael and daughter Jitka (b) Jaroslav Heyrovský with his colleagues in the Polarography Institute.

— Heyrovský noted its value and he added that the index rhythmically oscillates with the falling of the mercury drops. Then, by increasing voltage, the current further increased and, within the region of −1.9 V to −2.0 V, it suddenly started to strongly increase! To Heyrovský, it was undoubtedly clear that he made a first class discovery. He worked with enormous intensity and covered a 200-page laboratory notebook with his notes over the course of the next seven weeks. At that time, he also started writing the first publication on polarography (the term “polarography” was, however, introduced only later), which appeared in the 8th October 1922 issue of the Czech chemistry journal Chemické Listy. The title was “Electrolysis with the dropping mercury cathode”.

The polarographic device and analytical method

In order to speed up measurements with the dropping mercury electrode, J. Heyrovský with his Japanese coworker M. Shikata constructed an automatic apparatus, which they called a “polarograph”, in 1924. The apparatus continuously changed the mutual polarity of both electrodes and at the same time, photographically recorded the current passing through the solution and the electrodes, as a function of the electric voltage on the electrodes. Steps, so-called waves, are observed on the resulting polarisation curve. The height of the waves is a measure of concentration of substances dissolved in the solution, and their position indicates their quality. Based on the instrument, this method of electrolysis of solutions by means of the dropping mercury electrode was named polarography.

If a substance, which can be analysed both in its reduced or oxidised form, is present in the analysed solution, a step increase of the current appears on the curve, which is called a polarographic wave. The position of this wave along the electric potential axis characterises the substance in the solution and its height indicates the amount of the substance present in the solution. In chemical terms, we can hence say that polarography allows a simultaneous quantitative and qualitative analysis of a substance in the solution. If the solution contains several substances at the same time, the polarographic curve shows the corresponding number of waves, where each one of them indicates the quantity of the individual components of the solution. In this way a new, elegant and simple method of chemical analysis was born, which in many respects surpassed all other analytical methods of that time.

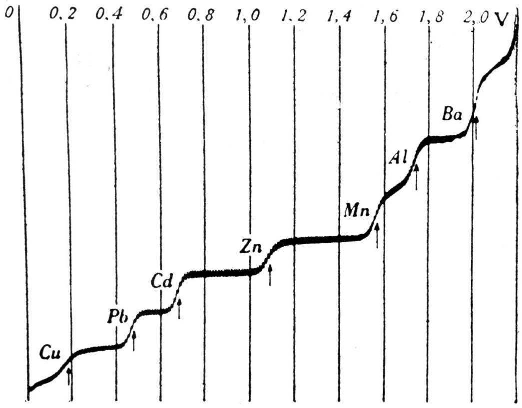

Figure 3: An example of polarographic curve

For a number of years, polarography was the "queen" among analytical methods thanks to both its high precision and a relatively low cost of polarographs. The Prague polarographic school formed by scientists from the whole of Europe has made the method known all around the world. The Polarographic Institute was founded in 1950 in Prague and Jaroslav Heyrovský became its director. Polarographic analysis was used in many sectors of industry where chemical analysis was necessary, as well as in the fields of biology, pharmacy and medicine (e.g. for the diagnosis of cancer). Great applications were found for polarographic long-time analysers — these instruments automatically followed a certain amount of a particular compound in a continuous production stream. If set up appropriately, they could also automatically control the work of a production line. J. Heyrovský with his team at the Polarographic Institute continued to develop further polarographic methods (e.g. oscillopolarography) and polarographic instruments.

Figure 4: Nobel prize winner Jaroslav Heyrovský and the king of Sweden Gustav Adolf VI. — Stockholm (10th December 1959)

The first Czech to be awarded a Nobel prize

The Nobel prize is often given to a scientist several years after the discovery when its importance is appreciated around the world, but sometimes they have to wait several decades for a recognition. Professor Heyrovský was proposed for the Nobel prize several times, but only in 1959 the decision was made. In that year he was proposed by the Nobel prize laureates A.J.P. Martin, C.V. Raman as well as by several other scientists. The Nobel committee also turned to Slovak universities for a suggestion, who unanimously supported Heyrovský. The Nobel prize award as if suddenly restored Heyrovský's decreasing strength and vitality. Unfortunately, his health soon started to deteriorate. In the 1960s

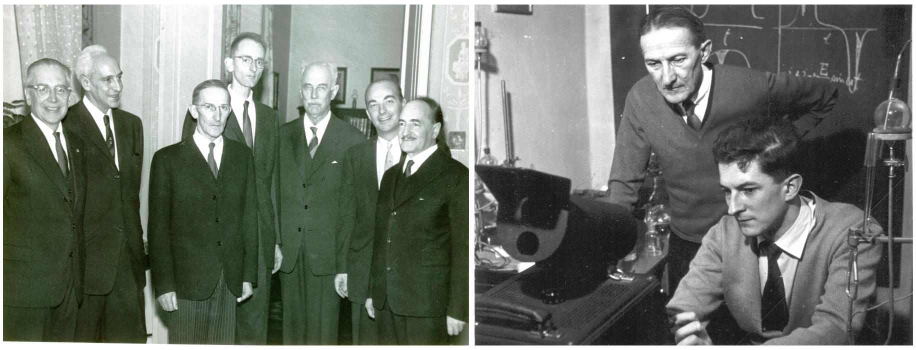

Figure 5: (a) Nobel prize laureates for the year 1959 (from left: E. Segre, E. Ochoa, J. Heyrovský, O. Chamberlain, chairman of the Nobel Committee B. Ekeberg, A. Kornberg, S. Quasimodo (b) Jaroslav and Michael Heyrovsky in laboratory.

Figure 6: (a) In 1958 the Czech polarographs shone at the world exhibition Expo 58. There was no polarograph that would not bring any awards from Brussels to Czechoslovakia, (b) the exhibition Story of the Drop in Zlín (March-April 2016)

he received a number of distinctions — for the second time he was honoured with the Order of the Czechoslovak Republic, he was elected as an honorary member of several academies and scientific societies, became a foreign member of the Royal Society, he received honorary doctorates of several European universities. However, due to his poor state of health he had to give up the leadership of the Polarographic Institute.

Heyrovský spent the last weeks of his life in the State Sanatorium in Smíchov, where he passed away on 27th March 1967. His last years resembled the end of life of Michael Faraday, his great role model, who died exactly hundred years earlier.

The travelling exhibition called “The story of a mercury drop” introduces Jaroslav Heyrovský to the young generations since the year 2009

Since 2009, which was the 50th anniversary of the award of the Nobel prize to Jaroslav Heyrovský on 10th December 1959, the wider public is reminded of this scientist by a travelling exhibition which presents the life and scientific work of Jaroslav Heyrovský not only as a reminder for those who are old enough to remember the age of polarography, but also for the younger generation — students and pupils.

The travelling exhibition “THE STORY OF A MERCURY DROP” which has since 2009 been exhibited at 31 places, can also be explored on the web page www.heyrovsky.cz.

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Using decomposition groups to prove theorems about quadratic residues

Perutka, Tomáš[1]

This paper introduces new proofs of some theorems about quadratic residues via decomposition groups and Frobenius automorphisms. What does it mean, though? The quadratic residues are quite important objects in number theory with many applications, both theoretical (solving quadratic congruences) and practical ones (in coding theory, acoustics etc.). Consider an odd prime p – that is, a natural number which is divisible only by 1 and itself (e.g. 3,5,11,...). We say that an integer a, which is not divisible by p, is a *quadratic residue modulo* p if its remainder after dividing by p is the same as the remainder of some perfect square (ie. c^2 where c is an integer). For example: -1 is a quadratic residue modulo 5 since it has the same remainder after dividing by 5 as $4 = 2^2$; 10 is a quadratic residue modulo 13 since it has the same remainder as $36 = 6^2$.

However, the quadratic residues and their properties are not the main focus of the paper, *the proofs* of the properties are. The most important tool used throughout the proofs is the *Frobenius automorphism*, a somewhat abstract and difficult concept from number theory. Its study has led to many breakthroughs, for example, to the proof of Fermat's Last Theorem. Therefore, it is not easy to explain the concept in short, however, let us attempt to do so.

We know that each integer can be decomposed uniquely into a product of prime numbers and the prime numbers cannot be decomposed any further (only as $1 \cdot p$ which is not interesting). However, this is no longer true if we move on to some larger set of numbers. For example, consider the set $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2}; a, b \text{ are the integers}\}$.

In this set, we obtain $7 = (3+\sqrt{2})(3-\sqrt{2})$, so we have just non-trivially decomposed a prime. This leads to a question: considering a set $\mathcal{O}$ similar as $\mathbb{Z}[\sqrt{2}]$ above (more specifically: the ring of integers of some finite field extension of the field of rational numbers), what can be said about how a prime p can be decomposed? This question has been one of the main questions in number theory for a long time. Frobenius automorphism of a prime p offers a partial answer. For any prime p and "a sufficiently nice" set $\mathcal{O}$, we always have a function $Frob_{p,\mathcal{O}}: \mathcal{O} \to \mathcal{O}$. Once the function is known, we know how p decomposes in $\mathcal{O}$. In particular, if $Frob_{p,\mathcal{O}}$ is an identity, p decomposes as much as possible. But how does all of this relate to the quadratic residues? Surprisingly, it can be shown that in the case $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, where $d \neq 0, 1$ is an integer , we get for any prime $p \neq 2$ the following: **$\boldsymbol{Frob_{p,\mathcal{O}}}$ is an identity if and only if $\boldsymbol{d}$ is a quadratic residue modulo $\boldsymbol{p}$.** We can then prove many marvelous theorems with this information and you can find out how in the paper.

__Tomáš Perutka__ is a student at the Faculty of Science in Masaryk University, Brno. He studies Pure Mathematics in the Department of Mathematics and Statistics. He is now mostly interested in algebraic number theory, category theory, and sheaf-theoretical methods in geometry and topology. He has been interested in number theory since high school; he has written there two manucripts about it which both competed in the competition SOČ. The first one won the first place in the competition and it also won an award from the Learned Society of the Czech Republic. Moreover, Tomas Perutka obtained an award "České hlavičky" and it was chosen to represent the Czech Republic at the international competition CASTIC in Macao where it won the bronze medal. The time Tomáš does not spend with mathematics he usually spends doing ballroom dancing or practising kung-fu.

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Using decomposition groups to prove theorems about quadratic residues

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In this text we elaborate on the modern viewpoint of the quadratic reciprocity law via methods of algebraic number theory and class field theory. We present original, short and simple proofs of so called additional quadratic reciprocity laws and of the multiplicativity of the Legendre symbol using decompositon groups of primes in quadratic and cyclotomic extensions of $\mathbb{Q}$.

Key words: Quadratic residue, decomposition group, Frobenius automorphism.

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Introduction

The quadratic reciprocity law is a theorem which enables us to work easily with quadratic residues. It was first proved by Gauss and since then it is (along with Fermat's Last Theorem) one of the results crucial for a further development of algebraic number theory: the work of Kummer, Kronecker, Artin and others on algebraic number theory and class field theory was partially motivated by trying to find a more general reciprocity law which would include the quadratic one as a special case. This quest was fulfilled via the Artin reciprocity law (more details, for example, are in [1]).

Gauss himself proved the quadratic reciprocity law in eight different ways; today there are more than 300 proofs (the complete list in [2]). Why is it so? Firstly, the theorem is quite popular and significant in number theory, it can be used, for example, to solve quadratic diophantine equations. The other reason is that people were looking for a proof which promises generalisation. The proof we work with is one of those proofs: it uses the full power of arithmetical results for finite Galois extensions of $\mathbb{Q}$, and the careful examination of cyclotomic and quadratic fields as well.

Let us now talk more about the structure of this text. In the first section, we mention some important prerequisites and results from algebraic number theory, mostly following [3], chapters 2-4. In Section 2, we present a proof of the quadratic reciprocity law. In Section 3, we reprove other statements about the quadratic residues by similar methods.

Conventions

The finite field with q elements will be denoted as $\mathbb{F}_q$. We will denote the element of the group $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$ containing an integer a as $[a]_n$; hence both $c \in [a]_n, [c]_n = [a]_n$ mean $c \equiv a \pmod{n}$. The ideal of a ring R generated by elements $a_1, ..., a_n$ will be denoted $(a_1, ..., a_n)$ if the ring R is evident from the context and $a_1R + \cdots + a_nR$ otherwise. All rings are commutative and prime ideals are assumed to be nonzero.

1 The prerequisites

1.1 Quadratic residues

We say that $m \in \mathbb{Z}$ is a quadratic residue modulo $n \in \mathbb{N}$ if m, n are coprime and there is $x \in \mathbb{Z}$ such that $x^2 \equiv m \pmod{n}$. We are interested only in quadratic residues modulo an odd prime p. This leads to the convention of the Legendre symbol:

Definition 1. *Let p be an odd prime, $a \in \mathbb{Z}$. Then the Legendre symbol $\left(\frac{a}{p}\right)$ is defined as follows:*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue mod } p, \\ 0 & \text{if } p \mid a, \\ -1 & \text{otherwise.} \end{cases}$$

The Legendre symbol gives rise to a homomorphism of groups $\left(\frac{\cdot}{p}\right) : \mathbb{F}_p^* \to \{\pm 1\}$. To see this, one needs to prove its multiplicativity. We will do this in Section 3.

The following theorem enables us to work with Legendre symbol quite easily:

Theorem 1. *Let p, q be a distinct odd primes. Then*

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) \cdot (-1)^{\frac{(p-1)(q-1)}{4}}. \tag{1}$$

Furthermore, the following holds:

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}, \tag{2}$$

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}. \tag{3}$$

The statement (1) is known as the quadratic reciprocity law; (2), (3) are usually called the additional or supplementary (quadratic) reciprocity laws.

To unpack the formulas a little, we will also introduce a reformulation via residue classes:

Theorem 1. *Let p, q be distinct odd primes. Then*

$$\left(\frac{p}{q}\right) = \begin{cases} \left(\frac{q}{p}\right) & \text{if } p \equiv 1 \text{ or } q \equiv 1 \pmod{4}, \\ -\left(\frac{q}{p}\right) & \text{if } p \equiv q \equiv 3 \pmod{4}. \end{cases}$$

Furthermore, the following holds:

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ -1 & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{8}, \\ -1 & \text{if } p \equiv \pm 3 \pmod{8}. \end{cases}$$

1.2 Galois extensions

We now very briefly recall the notion of a Galois extension (more details can be found for example in [4], Chapter VI). We say that the finite field extension K/k of degree n is Galois if there are precisely n k-automorphisms (i.e. they are trivial when restricted to k) of the field K. The group of k-automorphisms is then called the Galois group and denoted as $\mathrm{Gal}\,(K/k)$.

For example, any extension of finite fields is Galois with a cyclic Galois group. The case of our interest is mostly the case of Galois extensions $K/\mathbb{Q}$.

For Galois extensions, there is a one-to-one correspondence between the intermediate fields $k \subseteq M \subseteq K$ and the subgroups of $\mathrm{Gal}\,(K/k)$ realized as follows:

- $k \subseteq M \subseteq K \mapsto \{\sigma \in \mathrm{Gal}\,(K/k)\,; \sigma|_M = \mathrm{id}_M\}$,
- $H \subseteq \mathrm{Gal}\,(K/k) \mapsto \{\alpha \in K; \forall \sigma \in H : \sigma(\alpha) = \alpha\}$.

Moreover, the extension K/M is Galois for any intermediate field M. The extension M/k is Galois if and only if the group $\mathrm{Gal}\,(K/M)$ is a normal subgroup of $\mathrm{Gal}\,(K/k)$; in that case we have the isomorphism $\mathrm{Gal}\,(M/k) \cong \mathrm{Gal}\,(K/k) / \mathrm{Gal}\,(K/M)$ given by restriction to M. More precisely, the assignement $\sigma \in \mathrm{Gal}\,(K/k) \mapsto \sigma|_M \in \mathrm{Gal}\,(M/k)$ is a group homomorphism with kernel $\mathrm{Gal}\,(K/M)$.

1.3 Decomposition of primes in number fields

We will recall some basic facts from algebraic number theory. Consider any finite field extension $K/\mathbb{Q}$; by finiteness, each element of K is algebraic over $\mathbb{Q}$. Such K will be called a number field. We say that $\alpha \in K$ is an algebraic integer if its minimal polynomial over $\mathbb{Q}$ is monic with integer coefficients. Algebraic integers of K form a ring denoted as $\mathcal{O}_K$. It is well known that $\mathcal{O}_K$ is a Dedekind ring and hence each of its ideals (apart from $\{0\}$ and $\mathcal{O}_K$) has a unique decomposition as a product of prime ideals.

Consider any prime $p \in \mathbb{Z}$. We get

$$p\,\mathcal{O}_K = \mathcal{P}_1^{e_1} \cdots \mathcal{P}_g^{e_g},$$

where $\mathcal{P}_1, \dots \mathcal{P}_g$ are distinct prime ideals of $\mathcal{O}_K$; we say that they lie over p. Moreover, each field $\mathcal{O}_K / \mathcal{P}_i$ is a finite field of characteristic p with p^{f_i} elements. All these constants are neatly tied together:

$$\sum_{i=1}^{g} e_i f_i = [K : \mathbb{Q}].$$

This whole situation is much simpler if $K/\mathbb{Q}$ is Galois. In that case we get $e_1 = \cdots = e_g = e, f_1 = \cdots = f_g = f$ and hence $efg = [K : \mathbb{Q}]$.

We will end this section with an important observation about the decomposition of primes in the case that $\mathcal{O}_K = \mathbb{Z}[\omega]$ for some algebraic integer $\omega \in K$: this will occur in the setting of both quadratic and cyclotomic fields.

Theorem 2. *Let $\omega \in K$ be an algebraic integer such that $\mathcal{O}_K = \mathbb{Z}[\omega]$ and denote by $f(x) \in \mathbb{Z}[x]$ its minimal polynomial over $\mathbb{Q}$. Consider any prime $p \in \mathbb{Z}$. Let*

$$f(x) \equiv p_1(x)^{e_1} \cdots p_g(x)^{e_g} \pmod{p}$$

be the decomposition of $f(x) \mod p \in \mathbb{F}_p[x]$ as a product of monic irreducible polynomials. Then

$$p\,\mathcal{O}_K = \mathcal{P}_1^{e_1} \cdots \mathcal{P}_g^{e_g}$$

with $f_i = |\,\mathcal{O}_K / \mathcal{P}_i\,| = \deg p_i$; moreover, $\mathcal{P}_i = (p, \widetilde{p}_i(\omega))$ for any $\widetilde{p}_i \in \mathbb{Z}[x]$ such that $\widetilde{p}_i(x) \equiv p_i(x) \pmod{p}$.

Remark 2. This may appear as something very unintuitive and surprising, but that is not the case when seen through the lens of tensoring with $\mathbb{F}_p$. We already know that $p\,\mathcal{O}_K = \mathcal{Q}_1^{e'_1} \cdots \mathcal{Q}_{g'}^{e'_{g'}}$ for some prime ideals $\mathcal{Q}_i \subseteq \mathcal{O}_K$. So on one hand, we have

$$\mathbb{F}_p \otimes \mathcal{O}_K \cong \mathcal{O}_K / p\,\mathcal{O}_K \cong \prod_{i=1}^{g'} \mathcal{O}_K / \mathcal{Q}_i^{e_i'}$$

(the latter is due to the Chinese Reminder Theorem). And on the other hand,

$$\mathbb{F}_p \otimes \mathbb{Z}[\omega] \cong \mathbb{F}_p \otimes (\mathbb{Z}[x]/(f(x))) \cong \mathbb{F}_p[x]/(f(x) \mod p) \cong \prod_{i=1}^{g} \mathbb{F}_p[x]/\left(p_i(x)^{e_i}\right).$$

Although there are still some details to fill in (cf. [5], Thm. 4.12), the theorem above should now appear as a natural thing.

Remark 3. A more general version of the theorem is also true. If there is some algebraic integer $\omega \in K$ such that $\mathbb{Z}[\omega]$ is an additive subgroup of $\mathcal{O}_K$ of finite index d, the same conclusions as in the theorem above hold for each prime $p \in \mathbb{Z}$ not dividing d. This will come in handy in Section 1.5.

1.4 Arithmetic of abelian extensions of $\mathbb{Q}$

In this section we will work with number fields K such that $K/\mathbb{Q}$ is an abelian extension – i.e. Galois extension with commutative Galois group. We already know that for each prime $p \in \mathbb{Z}$ we have $p\,\mathcal{O}_K = \left(\mathcal{P}_1 \cdots \mathcal{P}_g\right)^e$ with $efg = [K : \mathbb{Q}]$, where $f = |\,\mathcal{O}_K / \mathcal{P}_i\,|$. Now we need to introduce some terminology:

Definition 4. *We say that the prime p*

- *ramifies in K if $e > 1$,*
- *totally ramifies in K if $e = n$ (and $f = g = 1$),*
- *is unramified in K if $e = 1$,*
- *is inert/remains prime in K if $f = n$ (and $e = g = 1$),*
- *splits completely in K if $g = n$ (and $e = f = 1$).*

The fact that K is a Galois extension enables us to define the following:

Definition 5. *Let $p \in \mathbb{Z}$ be a prime, $\mathcal{P} \subseteq \mathcal{O}_K$ be a prime ideal lying over p. Then the decomposition group of $\mathcal{P}$ in $K/\mathbb{Q}$ is the group*

$$D(\mathcal{P}) = \{\sigma \in \mathrm{Gal}(K/\mathbb{Q})\,;\sigma(\mathcal{P}) = \mathcal{P}\}.$$

Further, the inertia group of $\mathcal{P}$ in K is the group

$$I(\mathcal{P}) = \{\sigma \in \mathrm{Gal}(K/\mathbb{Q})\,;\forall \alpha \in \mathcal{O}_K : \sigma(\alpha) \equiv \alpha \pmod{\mathcal{P}}\}.$$

In the case of abelian extensions, the decomposition group does not depend on the prime ideal, but only on the prime it lies over: for any two $\mathcal{P}, \mathcal{P}'$ lying over p, we have $D(\mathcal{P}) = D\left(\mathcal{P}'\right), I(\mathcal{P}) = I\left(\mathcal{P}'\right)$. We thus speak about decomposition/inertia group of p and denote it as D_p, I_p; or $D_p(K), I_p(K)$ if we need to distinguish the field K. We can also see that $I_p \leq D_p$.

These groups carry much information about the decomposition of p. We can sum it up in the following proposition:

Proposition 6. *Let $\mathcal{P}$ be a prime lying over p. Denote $\widetilde{G} = \mathrm{Gal}\left((\mathcal{O}_K/\mathcal{P})/\mathbb{F}_p\right)$ – this is a cyclic group of order f generated by a Frobenius automorphism $\varphi : x \mapsto x^p$. Then, there is a canonical isomorphism $h : D_p/I_p \cong \widetilde{G}$. Moreover, the field $M_{D_p} \subseteq K$ fixed by D_p in Galois correspondence is the largest subfield of K in which p splits completely.*

Also, it can be shown that I_p is trivial precisely when p is unramified. In that case $D_p \cong \widetilde{G}$ is a cyclic group with a generator $\mathrm{Frob}_p = h^{-1}(\varphi)$. This generator is called the Frobenius automorphism of p.

The Frobenius automorphism of a prime is somehow a central notion of the algebraic number theory in Galois extensions. Moreover, it is a notion upon which class field theory has been built. Let us mention just two features of this automorphism:

Proposition 7. *Let K be an abelian number field, p a prime unramified in K. Then $Frob_p = \mathrm{id}$ if and only if p splits completely in K.*

Proposition 8. *Let K, k be abelian number fields with $k \subseteq K$. Then $Frob_p(K)|_k = Frob_p(k)$ for each prime $p \in \mathbb{Z}$.*

1.5 Quadratic fields

We define a quadratic field as an extension of $\mathbb{Q}$ of degree 2 – these are precisely the fields $\mathbb{Q}\left(\sqrt{d}\right)$ where $d \neq 0, 1$ is a square-free integer. Moreover, each such extension is abelian with $\mathrm{Gal}\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}\right) \cong \mathbb{Z}/2\mathbb{Z} \cong \{\mathrm{id}, \sigma_d\}$, where $\sigma_d\left(a + b\sqrt{d}\right) = a - b\sqrt{d}$.

There are only three ways in which a prime $p \in \mathbb{Z}$ can decompose in $\mathbb{Q}\left(\sqrt{d}\right)$:

- $e = 2, f = g = 1$ (totally ramifies),
- $f = 2, e = g = 1$ (remains prime),
- $g = 2, e = f = 1$ (splits completely).

It is straightforward to compute that $\mathcal{O}_{\mathbb{Q}\left(\sqrt{d}\right)}$ is equal to $\mathbb{Z}[\sqrt{d}]$ for $d \equiv 2, 3 \pmod 4$ and to $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ for $d \equiv 1 \pmod 4$. Hence, we know from Theorem 2 that in case $d \not\equiv 1 \pmod 4$, the decomposition of a prime $p \in \mathbb{Z}$ depends on the decomposition of $x^2 - d \pmod p$, i.e. on the Legendre symbol $\left(\frac{d}{p}\right)$!. Since $|\mathbb{Z}[\frac{1+\sqrt{d}}{2}]/\mathbb{Z}[\sqrt{d}]| = 2$, we see from Remark 3 that the same is true for odd primes in case $d \equiv 1 \pmod 4$. We thus arrive at the following proposition:

Proposition 9. *Let $d \neq 0, 1$ be a squarefree integer, $p \in \mathbb{Z}$ an odd prime. Then p*

- *totally ramifies as $\left(p, \sqrt{d}\right)^2$ if $\left(\frac{d}{p}\right) = 0$,*
- *splits completely as $\left(p, c - \sqrt{d}\right)\left(p, c + \sqrt{d}\right)$ if $\left(\frac{d}{p}\right) = 1, d \equiv c^2 \pmod p$,*
- *remains prime if $\left(\frac{d}{p}\right) = -1$.*

1.6 Cyclotomic fields

We define the n-th cyclotomic field, $n \in \mathbb{N}$, as $\mathbb{Q}(\zeta_n)$, where $\zeta_n = e^{\frac{2\pi i}{n}}$ is the primitive n-th root of unity. Again, these are all abelian extensions of $\mathbb{Q}$ with canonical isomorphism $\phi : (\mathbb{Z}/n\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. Since each automorphism of $\mathbb{Q}(\zeta_n)$ is uniquely determined by the image of ζ_n, we can define $\phi([a]_n)$ via $\zeta_n \mapsto \zeta_n^a$. This is easily seen to be well-defined and it can be proved that it really is an isomorphism.

Not only does the isomorphism ϕ work very naturally, it also behaves well with arithmetic properties:

Theorem 3. *Let $p \in \mathbb{Z}$ be a prime, $n \in \mathbb{N}$. Then p ramifies in $\mathbb{Q}(\zeta_n)$ if and only if $p \mid n$. For $p \nmid n$, consider the Frobenius automorphism $Frob_p \in D_p(\mathbb{Q}(\zeta_n))$ of p in the n-th cyclotomic field. Then $\phi([p]_n) = Frob_p$.*

This is a quite handy result, we can, for example, deduce the following:

Corollary 10. *A prime $p \in \mathbb{Z}$ splits completely in $\mathbb{Q}(\zeta_n)$ if and only if $p \equiv 1 \pmod n$.*

Proof. This follows from Prop. 7 and the theorem above. □

2 The proof of the quadratic reciprocity law

We start with the following lemma:

Lemma 11. *Let $p \in \mathbb{Z}$ be an odd prime. Then the only quadratic subfield of $\mathbb{Q}(\zeta_p)$ is of the form $\mathbb{Q}\left(\sqrt{p^*}\right)$, where $p^* = (-1)^{\frac{p-1}{2}} p$ is equal to p if $p \equiv 1 \pmod 4$ and to $-p$ if $p \equiv 3 \pmod 4$.*

Proof. It is well-known that $\left(\mathbb{F}_p^*\right)^2 = \{a^2 | a \in \mathbb{F}_p^*\}$ is the only subgroup of $\mathbb{F}_p^*$ of index 2. Since $\mathrm{Gal}\left(\mathbb{Q}(\zeta_p)/\mathbb{Q}\right) \cong \mathbb{F}_p^*$, we know from Galois theory that $\mathbb{Q}(\zeta_p)$ has a subfield K with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{F}_p^*/\left(\mathbb{F}_p^*\right)^2$, $\mathrm{Gal}\left(\mathbb{Q}(\zeta_p)/K\right) \cong \left(\mathbb{F}_p^*\right)^2$, $[K : \mathbb{Q}] = 2$, i.e. a quadratic subfield. We know from Theorem 3 that the only prime which ramifies in $\mathbb{Q}(\zeta_p)$ is p, hence we can conclude from Prop. 9 that $K = \mathbb{Q}\left(\sqrt{\pm p}\right)$. Now it is somewhat delicate to figure out the sign. One way to proceed is to use Gauss sums, the other way is to explore the decomposition of the prime 2. It is possible to show that 2 ramifies in $\mathbb{Q}\left(\sqrt{d}\right)$ for $d \not\equiv 1$ (and does not ramify otherwise) and hence we get $\pm p \equiv 1 \pmod 4$ and the lemma follows. □

Lemma 12. *The quadratic reciprocity law is equivalent to*

$$\left(\frac{q}{p}\right) = \left(\frac{p^*}{q}\right).$$

The proof is very easy and left to the reader as an exercise. □

We will now prove the quadratic reciprocity law by proving $\left(\frac{q}{p}\right) = 1 \Leftrightarrow \left(\frac{p^*}{q}\right) = 1$. The equality $\left(\frac{q}{p}\right) = 1$ is equivalent to the existence of an integer a such that $q \equiv a^2 \pmod p$, which is the same as $[q]_p \in \left(\mathbb{F}_p^*\right)^2$. But from the proof of Lemma 11 and from Theorem 3, we see that this is equivalent to $\mathrm{Frob}_q\left(\mathbb{Q}(\zeta_p)\right) \in \mathrm{Gal}\left(\mathbb{Q}(\zeta_p)/\mathbb{Q}\left(\sqrt{p^*}\right)\right)$. Now we use Prop. 8 to get $\mathrm{Frob}_q\left(\mathbb{Q}\left(\sqrt{p^*}\right)\right) = \mathrm{Frob}_q\left(\mathbb{Q}(\zeta_p)\right)|_{\mathbb{Q}(\sqrt{p^*})} = \mathrm{id}$ since the restriction from $\mathrm{Gal}\left(\mathbb{Q}(\zeta_p)/\mathbb{Q}\right)$ to $\mathrm{Gal}\left(\mathbb{Q}\left(\sqrt{p^*}\right)/\mathbb{Q}\right)$ corresponds to the projection $\mathbb{F}_p^* \to \mathbb{F}_p/\left(\mathbb{F}_p^*\right)^2$. This means that q splits completely in $\mathbb{Q}\left(\sqrt{p^*}\right)$, i.e. $\left(\frac{p^*}{q}\right) = 1$. Since every step is easily reversible, the quadratic reciprocity law is proved.

To put it in a concise way, the proof is actually the following chain of equivalences:

$$\begin{aligned}
\left(\frac{q}{p}\right) = 1 &\Leftrightarrow \exists a \in \mathbb{Z} : q \equiv a^2 \pmod p \\
&\Leftrightarrow [q]_p \in \left(\mathbb{F}_p^*\right)^2 \\
&\Leftrightarrow \mathrm{Frob}_q\left(\mathbb{Q}(\zeta_p)\right) \in \mathrm{Gal}\left(\mathbb{Q}(\zeta_p)/\mathbb{Q}\left(\sqrt{p^*}\right)\right) \\
&\Leftrightarrow \mathrm{Frob}_q\left(\mathbb{Q}\left(\sqrt{p^*}\right)\right) = \mathrm{Frob}_q\left(\mathbb{Q}(\zeta_p)\right)|_{\mathbb{Q}(\sqrt{p^*})} = \mathrm{id} \\
&\Leftrightarrow q \text{ splits completely in } \mathbb{Q}\left(\sqrt{p^*}\right) \\
&\Leftrightarrow \left(\frac{p^*}{q}\right) = 1.
\end{aligned}$$

Remark 13. This proof is not easy to find in the literature; the author has found it only in [6].

3 Proofs of additional laws and other statements

We will now apply the arithmetic of quadratic and cyclotomic fields to reprove other statements about quadratic residues. Since the situation is easier than in the case of the quadratic reciprocity law, we will be able to prove theorems in more detail.

Theorem 4. *For any odd prime p, we have* $\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ -1 & \text{if } p \equiv 3 \pmod 4. \end{cases}$

Proof. We will use the fact that the field $\mathbb{Q}(i)$ is both quadratic and cyclotomic. It is the quadratic field $\mathbb{Q}\left(\sqrt{-1}\right)$, hence we know from Proposition 9 that p splits completely in it if and only if $\left(\frac{-1}{p}\right) = 1$. It is also the cyclotomic field $\mathbb{Q}(\zeta_4)$, so we see from Corollary 10 that p splits completely in it if and only if $p \equiv 1 \pmod 4$. Hence $\left(\frac{-1}{p}\right) = 1 \Leftrightarrow p \equiv 1 \pmod 4$ and the theorem follows. $\square$

Theorem 5. *For any odd prime p, we have* $\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod 8, \\ -1 & \text{if } p \equiv \pm 3 \pmod 8. \end{cases}$

Proof. The proof will be analogous to the previous one, but more subtle. We will now work with a field $\mathbb{Q}\left(i, \sqrt{2}\right)$, which is an abelian extension of $\mathbb{Q}$ with $\mathrm{Gal}\left(\mathbb{Q}\left(i,\sqrt{2}\right)/\mathbb{Q}\right) = \{\mathrm{id}, \sigma, \tau, \sigma\tau\}$, where σ is given by $\sqrt{2} \mapsto -\sqrt{2}, i \mapsto i$ and τ by $\sqrt{2} \mapsto \sqrt{2}, i \mapsto -i$. We see from Galois theory that the lattice of intermediate fields of $\mathbb{Q}\left(i,\sqrt{2}\right)/\mathbb{Q}$ looks like this:

$$\begin{tikzcd}
 & \mathbb{Q}\left(i,\sqrt{2}\right) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(i) \arrow[dr, dash] & \mathbb{Q}\left(\sqrt{-2}\right) \arrow[d, dash] & \mathbb{Q}\left(\sqrt{2}\right) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Furthermore, $\mathbb{Q}\left(i,\sqrt{2}\right)$ is also the cyclotomic field $\mathbb{Q}(\zeta_8) = \mathbb{Q}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$, hence $\mathrm{Gal}\left(\mathbb{Q}\left(i,\sqrt{2}\right)/\mathbb{Q}\right) \cong (\mathbb{Z}/8\mathbb{Z})^{\times} = \{[1]_8, [3]_8, [5]_8, [7]_8\}$.

We thus have two explicit descriptions of the Galois group and now we have to find out how they match each other. For that, we need to compute how σ and τ act on ζ_8 (since they generate the Galois group):

- $\sigma\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} = \zeta_8^5 \Rightarrow \sigma$ corresponds to $[5]_8$,
- $\tau\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} = \zeta_8^7 \Rightarrow \tau$ corresponds to $[7]_8$.

Now, we already know well that $\left(\frac{2}{p}\right) = 1 \Leftrightarrow p$ splits completely in $\mathbb{Q}\left(\sqrt{2}\right) \Leftrightarrow \mathrm{Frob}_p\left(\mathbb{Q}\left(\sqrt{2}\right)\right) = \mathrm{id}$. But we also know that $\mathrm{Frob}_p\left(\mathbb{Q}\left(\sqrt{2}\right)\right) = \mathrm{Frob}_p\left(\mathbb{Q}\left(i,\sqrt{2}\right)\right)|_{\mathbb{Q}(\sqrt{2})}$, so $\left(\frac{2}{p}\right) = 1$ if and only if $\mathrm{Frob}_p\left(\mathbb{Q}\left(i,\sqrt{2}\right)\right) \in \mathrm{Gal}\left(\mathbb{Q}\left(i,\sqrt{2}\right)/\mathbb{Q}\left(\sqrt{2}\right)\right) = \{\mathrm{id}, \tau\} \cong \{[1]_8, [7]_8\}$ since τ fixes $\mathbb{Q}\left(\sqrt{2}\right)$ and corresponds to $[7]_8$. Hence $\left(\frac{2}{p}\right) = 1 \Leftrightarrow \mathrm{Frob}_p\left(\mathbb{Q}\left(i,\sqrt{2}\right)\right) \in \{\mathrm{id}, \tau\} \Leftrightarrow p \equiv 1, 7 \pmod 8$ and that is what we wanted to prove. $\square$

Theorem 6. *The Legendre symbol is multiplicative, i.e.* $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \cdot \left(\frac{b}{p}\right)$ *for each odd prime p and $a, b \in \mathbb{Z}$.*

Proof. The only non-trivial part of the proof is when we assume that a, b are distinct squarefree integers with, $a, b \neq 0, 1$, $p \nmid ab$. We will thus treat only this case and leave the rest of the proof to the reader.

With this assumption we see that $\mathbb{Q}\left(\sqrt{a}\right), \mathbb{Q}\left(\sqrt{b}\right), \mathbb{Q}\left(\sqrt{ab}\right)$ are quadratic fields embedded in the field $\mathbb{Q}\left(\sqrt{a}, \sqrt{b}\right)$ which is abelian of degree 4 over $\mathbb{Q}$:

$$\begin{tikzcd}
 & \mathbb{Q}\left(\sqrt{a},\sqrt{b}\right) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}\left(\sqrt{a}\right) \arrow[dr, dash] & \mathbb{Q}\left(\sqrt{ab}\right) \arrow[d, dash] & \mathbb{Q}\left(\sqrt{b}\right) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

The Galois theory tells us that these three quadratic fields are the only nontrivial subfields of $K = \mathbb{Q}\left(\sqrt{a}, \sqrt{b}\right)$. Consider the group $D_p(K)$. We have seen that this group fixes the largest subfield of K in which p splits completely. This subfield cannot be $\mathbb{Q}$ because then $D_p(K) = \mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ would not be cyclic (p cannot ramify because of our assumption $p \nmid ab$). If this subfield is K itself, p splits completely in all subfields and in particular, $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right) = 1$ and the theorem holds.

It remains to examine the case when $D_p(K)$ fixes one of the quadratic subfields. But in that case, p remains prime in the other two and the two of the three Legendre symbols are equal to -1 and the third to 1. The theorem hence follows. $\square$

Conclusion

The proofs we have seen in this text are just a few of many demonstrations of the fact that we have a very good understanding of the arithmetical theory of abelian extensions of $\mathbb{Q}$, which moreover behaves very nicely. The proofs of more general reciprocity laws over $\mathbb{Q}$ rely on class field theory, which is about much deeper examination of the phenomena we have seen here.

The natural question arises: can we prove in a similar way some kind of the quadratic reciprocity law over other number fields than $\mathbb{Q}$? It would certainly require a well-behaved arithmetical theory of abelian extensions of that field. In the current state of art, this is developed only for imaginary quadratic fields, i.e. fields $\mathbb{Q}\left(\sqrt{d}\right)$ with $d < 0$. It might be interesting to try and build a similar setting for the field $\mathbb{Q}(i)$, for example.

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Geomagnetic field mapping

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This work focuses on creating maps of the geomagnetic field and areas of increased cosmic radiation around the Earth. These areas must be mapped for a safe operation of satellites and to maintain the health of the crews on the International Space Station. This work was created in cooperation with the Institute of Technical and Experimental Physics at CTU which provided data measured by the Proba-V satellite in 2015. Proba-V is an ESA satellite with an almost circular trajectory at an altitude of about 810 km. One of the outcomes of this work are maps of the geomagnetic field distribution in the flight level of the satellite. The overall distribution of the geomagnetic field is shown by maps of the magnetic induction magnitude. Magnetic induction attains the lowest values in the area above the South Atlantic and South America and the highest values in the area south of Australia which is the area of the North Magnetic Pole. Other maps show the positions of the magnetic equator and the parallels. Geomagnetic field distribution maps for the individual months of 2015 show that the field has the same shape during the year. Another result was the mapping of areas of an increased number of charged particles around the Earth. These maps clearly show the position of the so-called South Atlantic anomaly and radiation belts around the magnetic poles. A comparison of geomagnetic and radiological maps showed that the area of the South Atlantic anomaly corresponds to the area of the lowest values of magnetic induction. The radiation bands around the poles copy the shape of the magnetic parallels. More focus was directed on the period around June 22, 2015, when a larger solar flare was observed around the Earth. The results of the measurements showed that at the time of the eruption, the number of charged particles around the Earth increased considerably and also that the areas where these particles accumulate significantly increased.

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Geomagnetic field mapping

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This work focuses on creating maps of the geomagnetic field and areas of increased cosmic radiation surrounding the Earth. Data were measured by Proba-V satellite at Low-Earth orbit 820 kilometres above the Earth during 2015. The actual measured data were compared with the calculated magnetic values. The created maps serve to a better understanding of the shape of the geomagnetic field and show magnetic equator, north magnetic pole and more. The map confirms that the area of the South Atlantic Anomaly corresponds with the weakest area of the geomagnetic field. Maps of different time periods of 2015 show small changes in the shape of the geomagnetic field during a year. Increased attention was paid to June 2015, when solar flares were passing near the Earth. The observation confirms that solar flares have a significant effect on the shape of the geomagnetic field.

Key words: Geomagnetic field, solar flare, ionizing radiation, South Atlantic Anomaly, SATRAM, Timepix, Proba-V, mapping.

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1 Introduction

One of the dangers that humanity must face while exploring the universe is an exposure to cosmic radiation. Cosmic radiation consists of high-energy, high-velocity ionizing particles that are constantly coming to Earth from the surrounding space. It has negative effects on human bodies and other living organism which are not adapted to live in areas with high ionizing radiation. The Earth's surface is protected from cosmic radiation by a dense atmosphere and a large magnetic field surrounding the Earth (hereinafter geomagnetic field). This two-component ”shield” can absorb or deflect almost all dangerous radiation from the Sun and from outer space. Ionizing particles coming from the Sun (so-called solar wind) are captured and diverted by the geomagnetic field. It causes the accumulation of ionizing radiation in certain areas around the Earth. These areas must be mapped for the safe operation of the satellites and also to maintain the health of the crews on the International Space Station (ISS).

The main goal of this work is to create detailed maps of the geomagnetic field at Low Earth orbit (LEO) and compare the results with known areas of increased ionizing radiation. The secondary goal is to observe the effect of solar flares on the geomagnetic field.

Geomagnetic field

The geomagnetic field extends tens of thousands of kilometers away from the planet. The main source of the geomagnetic field is the movement of a hot charged mass of the liquid Earth's core around the solid core. This mechanism is called geodynamo [1]. One of the external factors influencing the shape of the geomagnetic field are electric currents in the magnetosphere made by solar wind. This causes subsequent flattening of geomagnetic field on the side facing the Sun and stretching on the side facing away. The geomagnetic field is described by the magnetic dip I (or magnetic inclination) and the magnetic induction B. The magnetic dip is the angle of magnetic induction with respect to the horizontal plane. The magnetic induction on the earth's surface has a value of tens of μT while the direction and magnitude depend on geographical coordinates.

Figure 1 shows a simplified model of the geomagnetic field as an ideal bar magnet (red line). The axis of the geomagnetic field is 11 degrees away from the axis of rotation of the Earth. The center of the field is not the same as the center of the Earth but it is approximately 500 km closer to the region of Malaysia. Magnetic poles are not the same as geographic poles. The North Magnetic Pole is currently located in the Southern Hemisphere and vice versa. The position of the magnetic poles is constantly changing. The reversal of polarity of the field has occured several times already in the Earth's history.

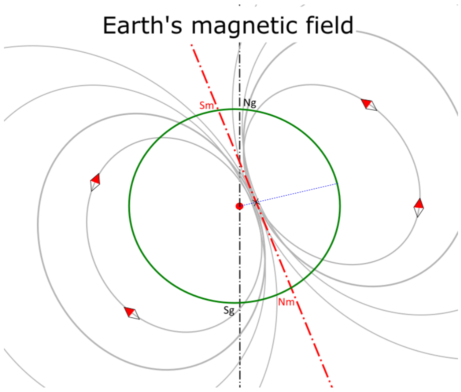

Figure 1: Simplified model of the geomagnetic field

Since 2013, the geomagnetic field has been studied by SWARM satellites [2] of the European Space Agency (ESA). SWARM consists of three satellites moving in the LEO area at an altitude of 450 km and 500 km above the surface.

Solar wind

Solar wind is a stream of high-energy particles arriving at Earth from the sun. It contains electrons, protons and alpha particles. Incoming particles are captured by the geomagnetic field, twisted along induction lines and accumulated in radiation belts in the Earth's magnetosphere (Van Allen belts [3]). The density of solar wind particles increases during solar flares. Increased solar activity can be observed on Earth in the form of aurorae. Large solar flares can cause reception disturbances on short radio waves and even power outages.

2 Experimental

Ground measurements of the geomagnetic field were performed using Magnetic Field Sensor and LabQuest 2 datalogger from Vernier. The equipment was borrowed from the Pardubice grammar school (Gymnázium Pardubice, Dašická 1083). Magnetic Field Sensor uses the Hall effect to measure magnetic induction. The sensor has two switchable measuring ranges. The first range is $\pm$ 0.3 mT with a sensitivity of 0.2 μT and the second range is $\pm$ 6.4 mT with a sensitivity of 4 μT. The range of $\pm$ 0.3 mT was set for the ground measurements. The measurement of the vertical component of the magnetic induction was performed by placing the Magnetic Field Sensor in a vertical position relative to the Earth's surface. The measurement of the horizontal component was performed by placing the sensor in a horizontal position relative to the Earth's surface and at the same time in the direction of the geomagnetic field line. The direction of the geomagnetic field line was determined by a compass at the measuring point. The total magnetic induction B was calculated from the measured values as $B = \sqrt{B_V^2 + B_H^2}$, where B_V is a vertical component of B and B_H is a horizontal component of B. Magnetic dip I was calculated as $I = \arccos \frac{B_H}{B}$. Two ground measurements were performed. The first measurement was performed in a family house in the city of Pardubice. The second measurement was performed in a field outside the city under an open sky. The second measurement was an attempt to avoid the influence of iron structures and power lines on the results.

Low-Earth orbit measurements were performed by Proba-V satellite. Proba-V was launched by ESA in May 2013 to study terrestrial vegetation. It is a small satellite with an almost circular trajectory at an altitude of 810 km to 822 km above the Earth's surface. The satellite's orbital period is 101 minutes. Proba-V is equipped with the Space Application of Timepix Radiation Monitor (SATRAM).

The Timepix sensor is used to detect and distinguish charged particles. The Timepix detector was delivered by the Institute of Experimental and Applied Physics (IEAP) of the Czech Technical University in Prague (CTU). Proba-V is equipped with multiple magnetic field sensors. Magnetic induction B is measured in three different axes relative to the Earth's surface and geographical coordinates. The first value of B is measured in the direction of flight of the satellite. The second value of B is measured in the direction to the center of the Earth's gravitational field. The remaining third value is perpendicular to the two previous ones. The total B is given as $B = \sqrt{B_1^2 + B_2^2 + B_3^2}$.

The maps created in this work were inspired by [4]. C. Granja (IEAP) created maps of dose rate (the ratio of the mean energy transmitted by ionizing radiation to a substance of mass) and described the South Atlantic anomaly (SAA) where inner Van Allen belt comes closest to the Earth's surface. Satellite data for the purposes of this work were provided by the IEAP. The maps in section 3 were created on the basis of data measured throughout 2015. The satellite performed more than four thousand measurements in a one day. The analyzed data consisted of approximately one and a half million measurements. The values displayed on the maps in the section 3 are the arithmetic averages of the values measured for the given coordinates for the given time period. The measured values of magnetic quantities were compared with the values calculated with [5].

3 Results

All ground measurements were performed at an altitude of 0.22 km in Pardubice. The measurement was performed as described in section 2. The measurement of the *city center* is the first measurement performed in a family house at the given coordinates. The second measurement was performed outside the city under the open sky. The values given in the Table 1 are equal to the arithmetic mean of the hundred-second measurement of B_H and B_V.

Table 1: Ground measurements

	City center [E15.792, N50.040]		Out of the city [E15.785, N50.068]	
	Measurement	Calculation	Measurement	Calculation
B_H [μT]	22.10	19.90	18.60	19.89
B_V [μT]	48.60	44.86	41.00	44.87
B [μT]	53.39	49.08	45.02	49.08
I [°]	65.54	66.07	65.60	66.09

B is magnetic induction and its components, I is magnetic dip

Figure 2 shows a map of the total B measured throughout 2015. It can be seen that the region with the highest B corresponds to the North magnetic pole. The area of lowest B is located above the area of South America and the South Atlantic. The map shows that the satellite can measure data above latitude from -81 to 81.

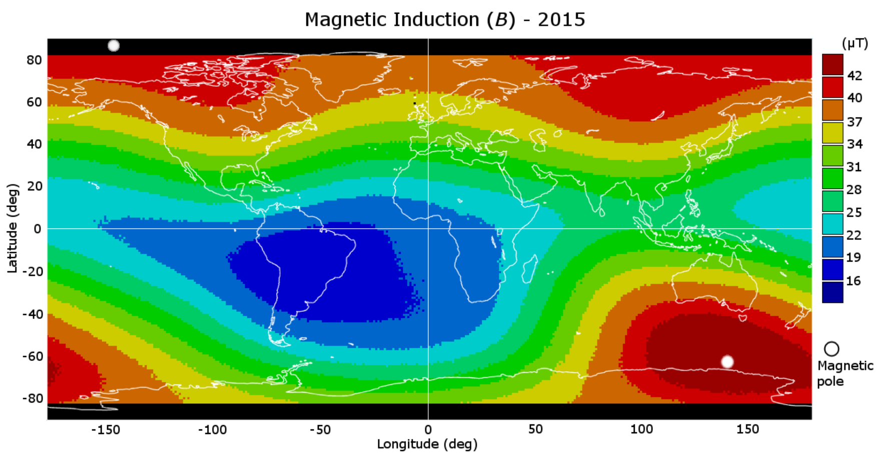

Figure 2: Map of the total B throughout 2015

Figure 3 shows maps of the total B measured in January, August and December 2015. It can be seen that the distribution of B above the Earth's surface is the same during these months.

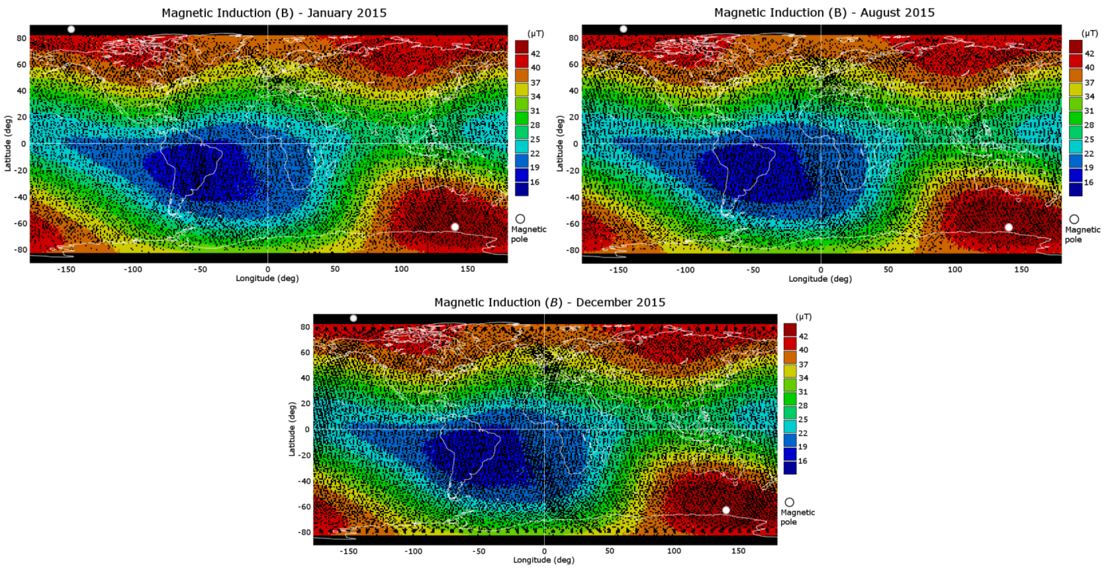

Figure 3: Maps of the total B in January (top left), August (top right) and December (bottom) 2015

The graph in Figure 4 shows the evolution of B measured by Proba-V above three different cities. The graph was compiled from data measured directly on the given coordinates, which caused that for each place only 8 to 11 measurements were performed during the year. Blue points indicate measurements above Brasilia (W47 S15, Brazil) located in the area with the lowest B values. Green points indicate measurements above Bangkok (E100 N13, Thailand) located in the area with intermediate B values. Red points indicate measurements above Hobart (E147 S42, Tasmania) located in the area with the highest B values. The lines indicate the calculated values for the given coordinates in the flight altitude of the Proba-V satellite.

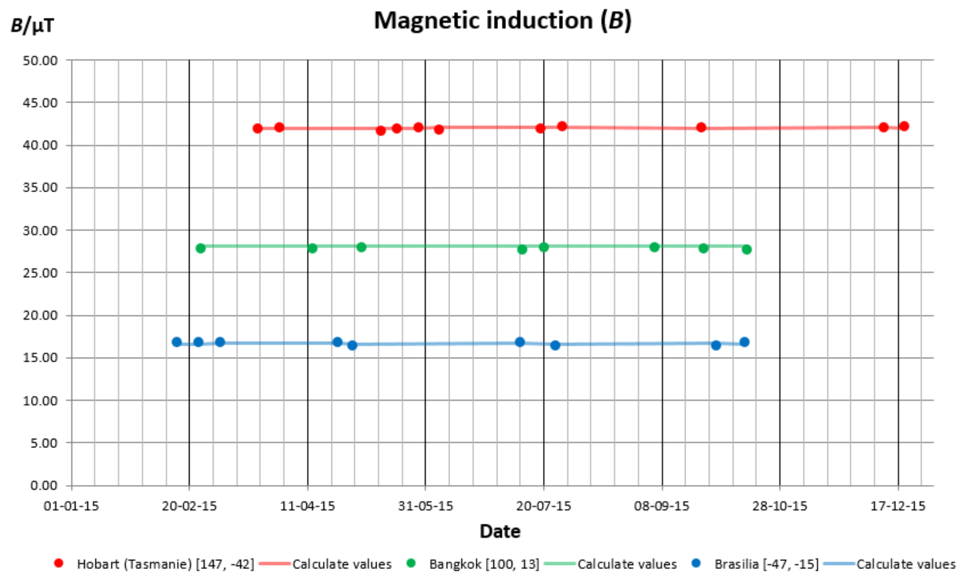

Figure 4: Evolution of magnetic induction over the selected cities during 2015

Figure 5 shows a map of the polarity of the vertical component of B measured throughout 2015. The boundary between negative and positive values indicates the magnetic equator. It can be seen that the magnetic equator does not correspond to the Earth's equator.

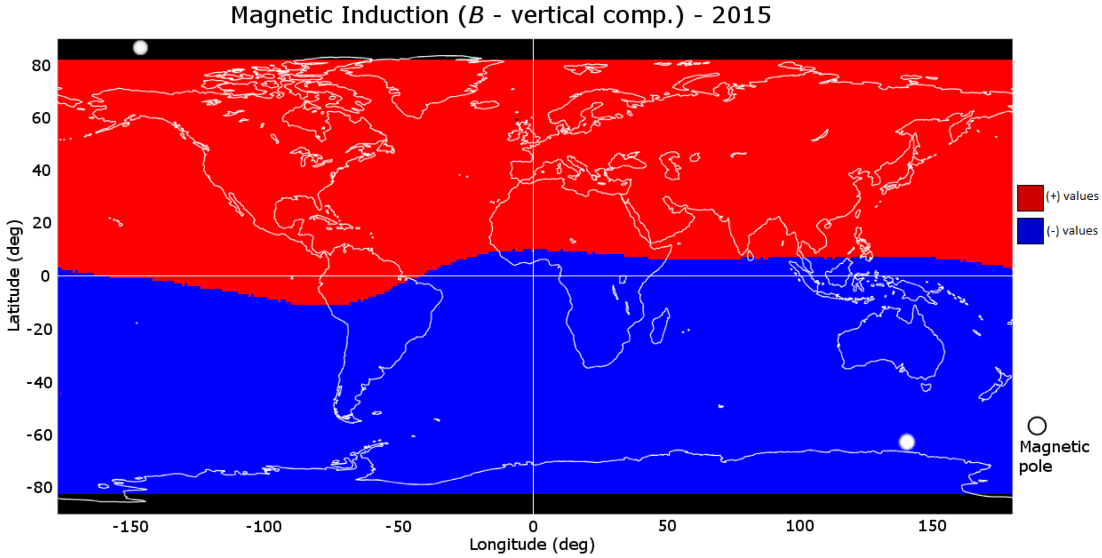

Figure 5: Magnetic equator

Figures 6 and 7 show maps of the vertical and the horizontal component of B measured in August 2015. One-month measurement was chosen due to less data distortion compared to a year-round measurement.

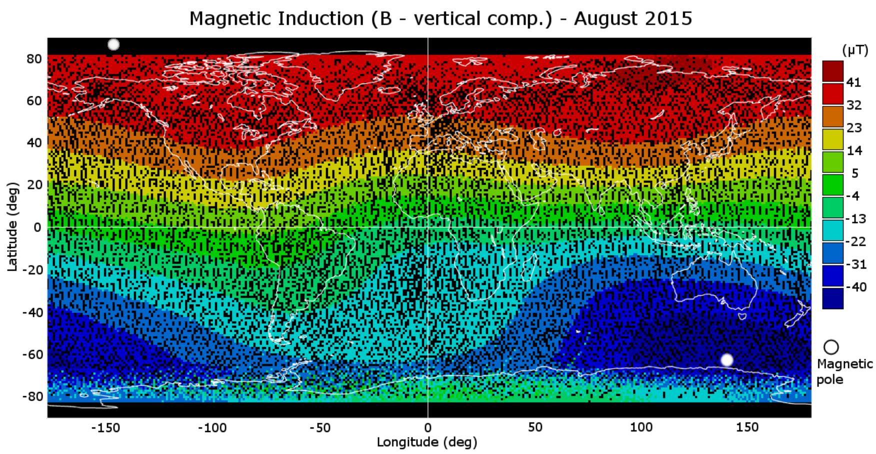

Figure 6: Map of the vertical component of B in August 2015

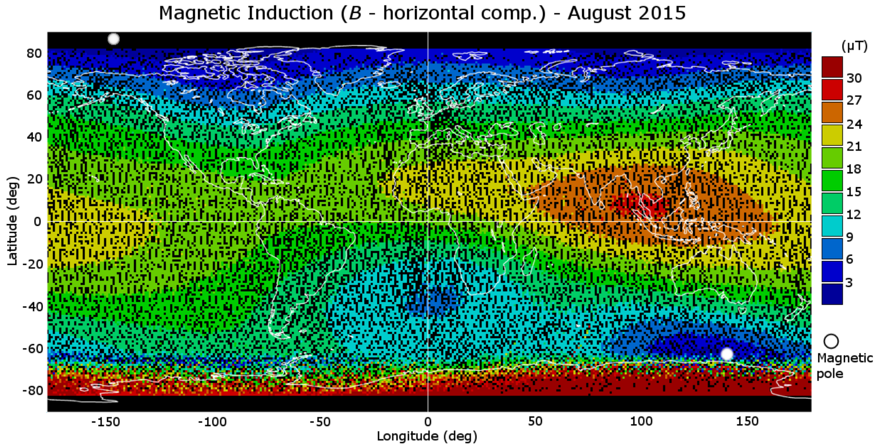

Figure 7: Map of the horizontal component of B in August 2015

Figure 8 shows a map of magnetic dip I (inclination) measured in June 2015. Figure 9 is a modified map 8 that shows magnetic circles of latitude (red points) and the magnetic equator (green points) using magnetic dip.

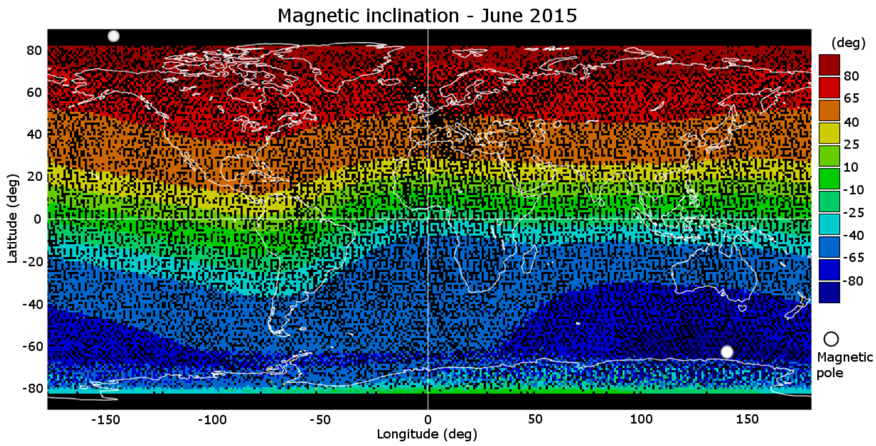

Figure 8: Map of magnetic dip in June 2015

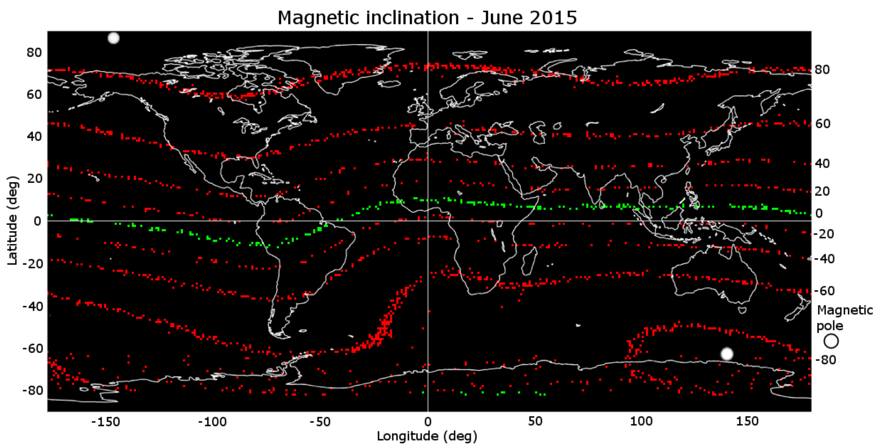

Figure 9: Map of magnetic circles of latitude in June 2015

Figure 10 shows a map of particle flux density (average number of charged particles per cm^2 per second) in June 2015 at the altitude of the Proba-V satellite. It can be seen that the highest particle flux density is in the area above South America and the South Atlantic and also in the belts around the polar regions.

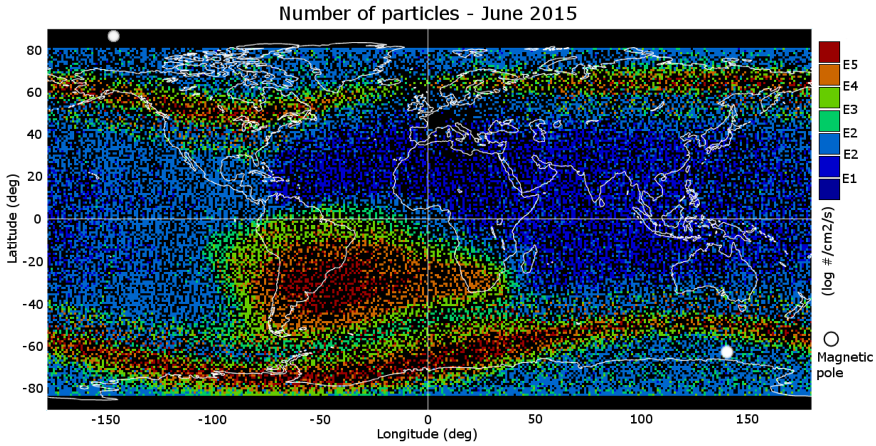

Figure 10: Map of particle flux density in June 2015

A large coronary mass ejection occurred on the surface of the Sun in the second half of June 2015. The results of this solar flare have been observable on Earth since June 22. The IEAP recommended comparing the three nine-day periods successively before (June 14 to June 22), during (June 23 to July 1) and after (July 2 to July 10) the manifestation of a solar flare near Earth. Dose rate maps for these three nine-day periods were created and provided by the IEAP. Figure 11 shows a comparison of the total B map (left) and the dose rate map (right) for for the period of the greatest manifestation of the solar flare. It can be seen that the area with the highest dose rate values and the lowest B values has increased during the solar flare.

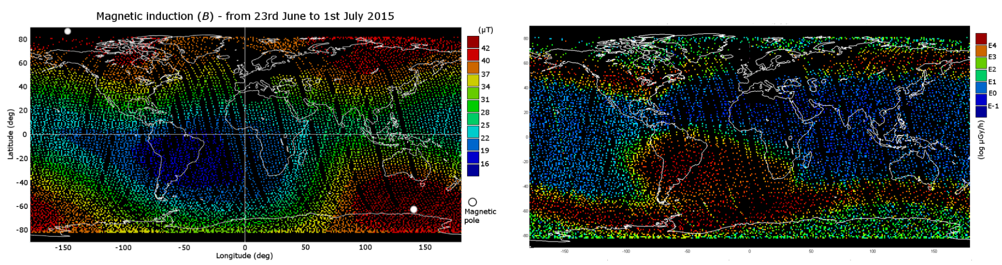

Figure 11: Comparison of the total B map and the dose rate map from June 23 to July 1

4 Discussion

Ground measurements from table 1 show that the magnetic induction at the surface of the Earth at coordinates E16 S50 is approximately 50 µT and the magnetic dip is approximately 65°. Satellite measurements at the same coordinates at the altitude of 820 km show that the magnetic induction is approximately 35 µT. The results thus confirm that the value of magnetic induction decreases with increasing altitude.

The map of total B during 2015 (Figure 2) shows that the distribution of the geomagnetic field depends on geographical coordinates. Satellite measurements show that magnetic induction has the highest values around coordinates E137 S64. This location has been designated the North Magnetic Pole. Satellite measurements show that magnetic induction reaches its lowest values in the area above South America and the South Atlantic. A comparison of the total B (Figure 2) and charged particle maps (Figure 10) shows that the area of the lowest total B values corresponds to the area with the highest charged particle occurrence called the South Atlantic Anomaly. The bands of charged particles around the polar areas (Figure 10) copy the shape of the magnetic parallels around the magnetic poles.

The measured properties of the geomagnetic field correspond with the description from section 1. The magnetic poles do not correspond to the geographical poles, the magnetic equator in Figure 5 is not identical with the geographic one, the center of the geomagnetic field is not the same as the center of the Earth, but is shifted towards the eastern and the northern hemisphere.

A comparison of the maps for the individual months of 2015 in Figure 3 showed that the geomagnetic field is not subject to any significant changes during the year. This was also confirmed by comparing measurements from three different locations above the Earth's surface, as shown in the graph in Figure 4. To achieve more accurate results, it would be necessary to focus on larger areas that would provide more measured data than shown in Figure 4.

Observations of the solar flare on June 22, 2015, showed that it was reflected in the close vicinity of the Earth by an increased number of charged particles. The maps in Figure 11 show that the areas of the highest charge particle occurrence have increased in size. The areas subsequently shrunk to their original state. The solar flare did not have a longer-term effect on the geomagnetic field.

5 Conclusion

The individual maps created in this work describe the distribution of the magnetic induction and its individual components during 2015. The results show that the geomagnetic field did not undergo any significant changes during 2015 and the distribution of the geomagnetic field is directly related to the occurrence of charged particles around the Earth. The observation of the solar flare from June 2015 show that it has no long-term effect on the shape of the geomagnetic field.

Acknowledgements

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Optimizing length of planar curves

Kloud, Vojtěch[1], Bednařík, Dušan[2]

The article focuses on the problem of finding a shortest path in plane with obstacles. Problems of such nature occur for instance in robotics or transport and are of great importance. Their optimal solutions can save not only time of travel but also fuel costs. Problem of finding a shortest path in plane with obstacles is usually treated as a discrete problem, meaning that the plane is thought of as a grid of points, between which one can move only horizontally, vertically or diagonally. Thus the problem loses its smooth nature and such approach may seem too restrictive.

Because of this we treat the problem purely analytically or continuously, which in turn allows one to follow any smooth path in the plane without too many restrictions. Because of the lack of literature concerning such continuous approach, basic definitions of continuous obstacles are given. Also, the set of all potential solutions to a problem is defined. This set is called the set of all admissible curves and the goal is to find the shortest one.

Basics of the continuous version of the problem are treaded using known methods of mathematical analysis and calculus of variations. After giving appropriate definitions, basic case of the problem with only one obstacle is given. There, we use proof by contradiction to show which curve is the shortest one under the assumption that there exist one such curve. We also prove the existence of a solution to this problem using some useful derived properties of the length of admissible curves. In mathematics, the assumption of existence of a solution is a strong one and therefore proof of such nature is of great significance.

Vojtěch Kloud *studies at První soukromé jazykové gymnázium in Hradec Králové. His main focus is the study of mathematics, more precisely mathematical analysis. He placed second with his thesis on optimal paths in the Czech nation round of Students professional activities - SPA (Středoškolská odborná činnost - SOČ) and became an Intel ISEF Finalist which was supposed to také place in May 2020 in Anaheim (USA). He was also rewarded for his thesis by an award from the Learned Society of the Czech Republic. Besides academic interests, Vojtěch competes as an athlete on international level in pole vault and decathlon, and is a former U18 national champion in pole vault. His plan is to attend a college in the STEM field in the United States or the Czech Republic while continuing to pursue his interests in athletics and music. This presented paper is an extract from his 2019 thesis of the same title.*

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Optimizing length of planar curves

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This article focuses on the problem of finding a shortest path in plane with obstacles. Problems of such nature occur for instance in robotics or transport and are of great importance. The problem is analyzed using the methods of mathematical analysis and calculus of variations. Definitions of basic concepts of the problem are given. From these definitions, useful properties, such as convexity of the length functional, are proven. These properties are used to show the existence of a solution in one of the considered cases of the problem. Other case of the problem was considered, where it is established under which conditions does a shortest path attain its general form and what this form looks like.

Key words: Optimization, calculus of variations, planar curves, constraints, obstacles.

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1 Introduction

The main goal of this paper is to answer a seemingly trivial question; what is the shortest path in a plane between two endpoints that avoids given obstacles? People usually have a strong intuition for finding a solution to such problems, and this will be supported with mathematically rigorous methods.

Problems of shortest paths are usually treated as discrete problems in graph theory. Practically, this approach is good enough but some conventional algorithms (A* algorithm, Dijkstra's algorithm) for instance do not guarantee that the chosen path is indeed the shortest.[1]

Because of this, the problem will be treated purely analytically so that different, hopefully more effective methods for solving the problem at hand, can be utilized. Due to the lack of literature concerning such approach, suitable definitions and notations involved in the problem will firstly be introduced, namely the idea of an obstacle and an admissible curve. From an analytical perspective, the problem can be further classified as a constrained variational problem.

2 Theoretical part

In order to develop an elementary understanding of the problem, calculus of variations along with theorems of real analysis will be studied. The concept of a functional and first variation will be introduced while omitting details that can be found in books on calculus of variations and real analysis.[2,3]

2.1 Calculus of variations

Many actions in nature can be quantified with an appropriate functional, usually given in an integral form. For example, a path described by a smooth function $y(x)$ on an interval $[a, b]$ is measured by $\int_a^b \sqrt{1 + (y')^2} dx$. This can intuitively be shown by slicing up the interval into small pieces and measuring the length of each small hypotenuse described by by the change in x, dx and the change in y, dy.

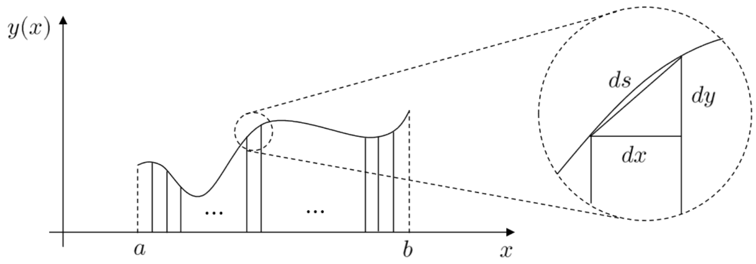

Figure 1: Understanding the length functional

The infinitesimal lengths ds can then be added up along the interval $[a, b]$ by taking the integral over $[a, b]$. Since dx goes to 0, for the derivative of $y(x)$ the equality $y' = \frac{dy}{dx}$ holds. Therefore:

$$L(y) = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + (y')^2} dx$$

Definition 2.1. *Suppose a function y has a continuous derivative on $[a, b]$. The length of y on $[a, b]$ is defined by the integral*

$$L(y) = \int_a^b \sqrt{1 + (y')^2} dx \tag{1}$$

Definition 2.2. *A functional is a function*

$$I : \mathcal{F} \ni y \to I(y) \in \mathbb{R},$$

where $\mathcal{F}$ is a space or a set of functions.

The natural question is then to ask what function minimizes the functional $L(y)$. Calculus of variations provides a necessary condition for an extreme of a functional in form of the first variance and the Euler-Lagrange differential equation.

Theorem 2.1. *If a function y is a local extreme of the functional $I(y)$ that is defined on $\mathcal{F}$ and h is a function such that $y + th \in \mathcal{F}$ for all small $t \in (-\varepsilon, \varepsilon)$, then*

$$\delta I(y)h = \lim_{t \to 0} \frac{I(y + th) - I(y)}{t} = 0.$$

The expression $\delta I(y)h$ is called the first variance of the functional I in y in the direction of h.

Theorem 2.2. *Suppose a functional I is of the form $I(y) = \int_a^b F(x, y, y')dx$, where F is continuously partially differentiable with respect to all there variables and y is a smooth function of x. If y is a local extreme of I and has fixed endpoints: $y(a) = A, y(b) = B$, then it must satisfy the Euler-Lagrange equation:*

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{2}$$

This is one of the most important theorems of calculus of variations. Proof will be omitted but can be found in books on the topic.[2] The equation can be applied to the length functional (1) to find the shortest path between the points $[a, A]$ and $[b, B]$. By taking the appropriate partial derivatives, the equation (2) becomes:

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = 0.$$

The solution to this differential equation is of the form $y(x) = c_1 x + c_0$ which with $c_1 = \frac{B-A}{b-a}$ and $c_0 = B - \frac{B-A}{b-a}$ is a straight line going through the desired points. With a little more effort, it can be shown that this solution is the global minimizer. Therefore, the shortest path between two points in a plane is a straight line. Many other functional can be formed to describe different aspects of a curve. For example, the functional $T(y) = \int_a^b \frac{\sqrt{1+(y')^2}}{\sqrt{y}} dx$ gives a total time in which a particle gets from $[a, A]$ to $[b, B]$ in a homogeneous gravitational field. The solution of the corresponding Euler-Lagrange equation then gives the path of shortest time - The Brachistochrone. Other examples along with solutions and with proofs of preceding theorems can be found in books on calculus of variations of authors paper.[2,4]

2.2 Theorems of real analysis

In order to support the intuition behind the original problem with mathematically rigorous methods, some of the most important theorems of real analysis will be utilised. Their proofs and other useful theorems can be found in books on mathematical analysis.[3]

Theorem 2.3 (Bolzano's theorem). *If a real function f is continuous on $[a, b]$ and satisfies $f(a)\cdot f(b) \leq 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.*

Theorem 2.4 (Mean Value Theorem). *If a function f is differentiable on $[a, b]$, then there exists $c \in [a, b]$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.*

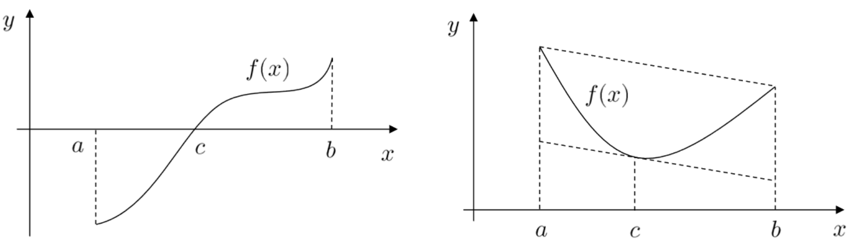

Figure 2: Bolzano's Theorem and Mean Value Theorem

These two theorems will be used as lemmas when proving that a given curve is indeed the shortest one. Since convexity and concavity play a role in the problem, definition of convex set and concave function will now be given.

Definition 2.3. *Suppose X is a subset of a vector space E. The set X is called convex if $\forall x, y \in X \wedge t \in [0, 1]$ the line segment $tx + (1 - t)y \in X$.*

Definition 2.4. *A real function f is said to be concave on an interval X, if $\forall x_1, x_2 \in X \wedge \forall t \in [0, 1]$ the inequality*

$$f(tx_1 + (1 - t)x_2) \leq tf(x_1) + (1 - t)f(x_2)$$

holds.

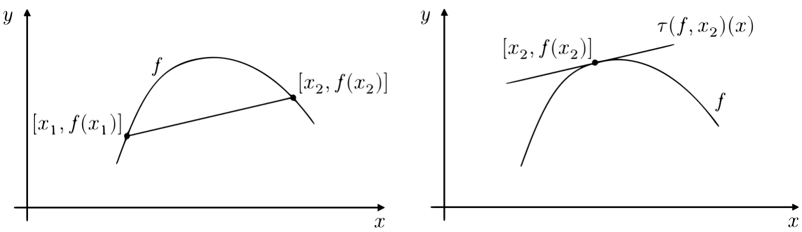

Figure 3: Properties of real concave functions

From this definition it can additionally be shown that the tangents to a concave function lie above the function, the first derivative of a concave function is not increasing and its second derivative is non-positive.[5]

3 Results

Firstly, we will provide definitions of obstacles and admissible curves. It will later be shown that other useful properties can be derived from these definitions.

Definition 3.1. *An obstacle $\mathcal{P} \in \mathbb{R}^2$ is given by a pair $(p_1, >)$ or $(p_2, <)$, where p_1, p_2 are continuous functions. The obstacles themselves are defined as the following sets:*

$$\mathcal{P}(p_1, >) = \{[a_x, a_y] | a_y > p_1(a_x)\}$$

$$\mathcal{P}(p_2, <) = \{[b_x, b_y] | b_y < p_1(b_x)\}$$

The functions describing the obstacle may have a restricted range to a closed interval.

Definition 3.2. *Suppose desired endpoints $[a, A], [b, B]$ are given along with obstacles $\mathcal{P}_1, ..., \mathcal{P}_n$ on $[a, b]$. A differentiable curve y is said to be admissible if:*

$$\forall x \in [a, b], \forall m = 1, ..., n : [x, y(x)] \notin \mathcal{P}_m \wedge y(a) = A, y(b) = B. \tag{3}$$

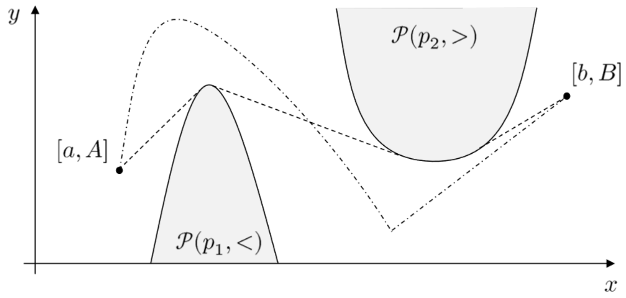

Figure 4: Visualisation of the problem

3.1 One obstacle with fixed endpoints

Suppose only one obstacle $\mathcal{P}(p, <)$ described by smooth concave function p is given along with two endpoints that lie on the obstacle: $p(a) = A$, $p(b) = B$. The set of all admissible curves $\mathcal{A}$ is of the form:

$$\mathcal{A} = \{y | y \text{ is differentiable}, y(x) \geq p(x) \forall x \in [a, b], y(a) = p(a), y(b) = y(p)\}. \tag{4}$$

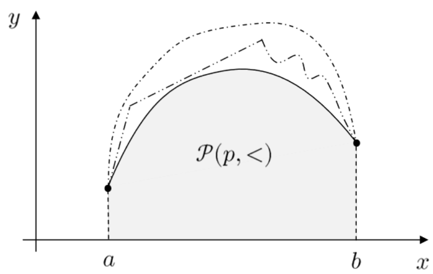

Figure 5: Problem with a concave function

Intuitively, the shortest path leads along the obstacle, that is, the curve $y = p$ is the shortest one.

Theorem 3.1. *Suppose we are given an obstacle $\mathcal{P}(p, <)$, where p is differentiable, and two endpoints $A = p(a), B = p(b)$. Under the assumption of an existence of a global minimizer, the path $y = p$ is the shortest one in $\mathcal{A}$.*

Proof. We proceed by contradiction. Suppose that shortest path satisfies $y(c) > p(c)$ for some $c \in (a, b)$. In this case we can define a new function $y_{pw}(y)$ the following way:

$$y_{pw}(x) = \begin{cases} y(x) & \text{for } x \in [a, e_a] \\ p'(c)(x - c) + p(c) & \text{for } x \in [e_a, e_b] \\ y(x) & \text{for } x \in [e_b, b] \end{cases}$$

, where e_a, e_b are points such that $p'(c)(e_a - c) + p(c) = y(e_a)$ and $p'(c)(e_b - c) + p(c) = y(e_b)$. Their existence is guaranteed by Bolzano's theorem 2.3 and the concavity of p. The length of the original curve y can be broken up into three integrals:

$$L(y) = \int_a^{e_a} \sqrt{1 + (y')^2} dx + \int_{e_a}^{e_b} \sqrt{1 + (y')^2} dx + \int_{e_b}^{b} \sqrt{1 + (y')^2} dx.$$

The length of y_{pw} can be evaluated from its definition:

$$L(y_{pw}) = \int_a^{e_a} \sqrt{1 + (y')^2} dx + \int_{e_a}^{e_b} \sqrt{1 + (p'(c))^2} dx + \int_{e_b}^{b} \sqrt{1 + (y')^2} dx.$$

Because $p'(c)(x - c) + p(c)$ is a straight line between $[e_a, y(e_b)]$ and $[e_b, y(e_b)]$ and y is not straight on $[e_a, e_b]$ by definition, we have $L(y_{pw}) < L(y)$, which is a contradiction, because y was assumed to be the shortest path. Therefore, if there is a minimizer, it must be p. This completes the proof. □

In this basic example, one can proceed without needing to assume the existence of a minimizer. Traditionally, direct methods of calculus of variations are used to prove statements of such nature. However, the following theorem is fairly elementary and does not require any background in these methods. On the other hand, it works with a stronger assumption on p.

Theorem 3.2. *Suppose we are given an obstacle $\mathcal{P}(p, <)$, where p is twice differentiable and two endpoints $A = p(a), B = p(b)$. The global minimizer is the path $y = p$.*

Proof. We wish to prove that for a concave function p on $[a, b]$ and a differentiable function $d(x) \geq 0$ on $[a, b]$ and $d(a) = d(b) = 0$, the inequality

$$\int_a^b \sqrt{1 + (p' + d')^2} dx \geq \int_a^b \sqrt{1 + (p')^2} dx$$

holds. Evidently, $(d')^2 \geq 0$ for any real function on $[a, b] \subset \mathbb{R}$. This is equivalent to

$$[1 + (p' + d')^2][1 + (p')^2] \geq [p'd' + 1 + (p')^2]^2$$

for any p'. Since both sides of the inequality are positive, square root on both sides can be taken and the inequality sign kept:

$$\sqrt{1 + (p' + d')^2}\sqrt{1 + (p')^2} \geq p'd' + 1 + (p')^2$$

After dividing both sides by $\sqrt{1 + (p')^2}$ and rearranging, the inequality becomes

$$\sqrt{1 + (p' + d')^2} - \sqrt{1 + (p')^2} \geq \frac{p'd'}{\sqrt{1 + (p')^2}} \tag{5}$$

By integrating both sides on $[a, b]$ we get the following integral inequality:

$$\int_a^b \sqrt{1 + (p' + d')^2} dx - \int_a^b \sqrt{1 + (p')^2} dx \geq \int_a^b \frac{p'd'}{\sqrt{1 + (p')^2}} dx.$$

On the right side we can proceed with integration by parts. Let

$$u = \frac{p'}{\sqrt{1+(p')^2}} \qquad v' = d'$$

$$u' = \frac{p''\sqrt{1+(p')^2} - \frac{(p')^2}{\sqrt{1+(p')^2}}}{1+(p')^2} \qquad v = d$$

$$\int_a^b \frac{p'd'}{\sqrt{1 + (p')^2}} dx = \left[\frac{p'}{\sqrt{1 + (p')^2}} \cdot d\right]\Bigg|_a^b + \int_a^b \left[\frac{\frac{(p')^2}{\sqrt{1+(p')^2}} - p''\sqrt{1 + (p')^2}}{1 + (p')^2} \cdot d\right] dx$$

The first summand on the right side vanishes since $d(a) = d(b) = 0$. Since p is concave, then $-p'' \geq 0$. This means that the whole integrand is positive, therefore, the whole integral must be positive, finally giving us the desired inequality:

$$\int_a^b \sqrt{1 + (p' + d')^2} dx - \int_a^b \sqrt{1 + (p')^2} dx \geq \int_a^b \frac{p'd'}{\sqrt{1 + (p')^2}} dx \geq 0$$

$$\int_a^b \sqrt{1 + (p' + d')^2} dx \geq \int_a^b \sqrt{1 + (p')^2} .dx$$

□

3.2 Convexity of $\mathcal{A}$ and the functional L

Proving that a given set or a function is convex is useful, because one can consequently resort to more methods of solving a given problem using already developed methods of convex optimization.[5] In this chapter, we will demonstrate the convexity of the set $\mathcal{A}$ defined by (3) and the length functional $\int_a^b \sqrt{1 + (y')^2} dx$.

Theorem 3.3. *Let $C^1[a, b]$ be the set of all differentiable functions on $[a, b]$. The subset $\mathcal{A} \subset C^1[a, b]$ defined by (3) is a convex set.*

Proof. Without loss of generality, we will only prove that the set (4) is convex. Let $y_1, y_2 \in \mathcal{A}$. Then the function

$$y(x) = ty_1(x) + (1 - t)y_2(x),$$

where $x \in [a, b] \wedge t \in [0, 1]$. y is differentiable, since y_1, y_2 are differentiable and t is a constant with respect to x. Since $t \geq 0$ and $1 - t \geq 0$ the two inequalities $ty_1(x) \geq tp(x), (1 - t)y_2(x) \geq (1 - t)p(x)$ can be added to obtain

$$y(x) = ty_1(x) + (1 - t)y_2(x) \geq tp(x) + (1 - t)p(x) = p(x).$$

Also, $y_1(a) = y_2(a) = p(a)$ and $y_1(b) = y_2(b) = p(b)$ and therefore

$$y(a) = ty_1(a) + (1 - t)y_2(a) = tp(a) + p(a) - tp(a) = p(a),$$

$$y(b) = ty_1(b) + (1 - t)y_2(b) = tp(b) + p(b) - tp(b) = p(b).$$

The function satisfies all three conditions in (4) and therefore $y(x) = ty_1(x)+(1-t)y_2(x) \in \mathcal{A} \quad \forall y_1, y_2 \in \mathcal{A} \wedge t \in [0, 1]$, making the set $\mathcal{A}$ convex. □

Theorem 3.4. *The length functional $I(y) = \int_a^b F(x, y, y')dx$ is convex over all curves $\mathcal{A} \subset C^1[a, b]$.*

Proof. The functional $I(y) = \int_a^b F(x, y, y')dx$ is said to be convex, if it satisfies

$$F(x, y + h, y' + k) - F(x, y, y') \geq h\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial y'} \tag{6}$$

$\forall [x, y, y', h, k] \in [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.[2] For the integrand $F = \sqrt{1 + (y')^2}$, where $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}$ with $[h, k] = [d, d']$, the inequality (6) is equivalent to (5), making the length functional $L(y) = \int_a^b \sqrt{1 + (y')^2} dx$ convex. □

3.3 One obstacle with free endpoints

Let us have an obstacle $\mathcal{P}(p,<)$, where p is a concave, continuously differentiable on $[\alpha,\beta] \subset [a,b]$ and two endpoints $[a,A],[b,B]$. If the straight line $l_{AB}(x)$ between the two endpoints is admissible, then it is also the shortest path. If it is not, we suppose that l_{AB} and p intersect at points α,β such that $a<\alpha<\beta<b$.

Theorem 3.5. *Under the above mentioned assumptions, there exist two points c_a, c_b such that the curve $y \in \mathcal{A}$ of the form*

$$y(x)=\begin{cases} p'(c_a)(x-c_a)+p(c_a) & \text{for } x\in[a,c_a] \\ p(x) & \text{for } x\in[c_a,c_b] \\ p'(c_b)(x-c_b)+p(c_b) & \text{for } x\in[c_b,b] \end{cases},$$

where $y(a)=A, y(b)=B$ exists and is the shortest one in $\mathcal{A}$.

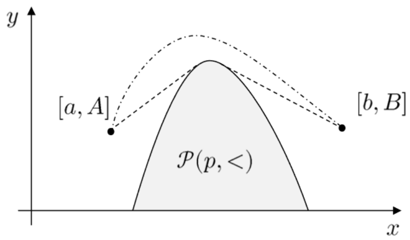

Figure 6: Second case of the problem

Proof. The existence of c_a and c_b is proven by the following argument. Without loss of generality, we only give proof of c_a's existence. If α and β exist, then by the Mean Value Theorem 2.4 there exists $c\in[\alpha,\beta]$ such that $p'(c)=\frac{p(\beta)-p(\alpha)}{\beta-\alpha}=\frac{B-A}{b-a}$, meaning that the tangent $p'(c)(x-c)+p(c)$ is parallel with l_{AB} and lies on or above l_{AB}, therefore $p'(c)(a-c)+p(a)\geq A$. The function p is concave, therefore p' is non-increasing, which means $p'(\alpha)\geq p'(c)=\frac{B-A}{b-a}$. Also, since $a-\alpha<0$, the following holds:

$$p'(\alpha)(a-\alpha)\leq\frac{B-A}{b-a}(a-\alpha).$$

After adding $p(\alpha)=l_{AB}(\alpha)=\frac{B-A}{b-a}(\alpha-a)+A$ to both sides if the inequality, we yield the property

$$p'(\alpha)(a-\alpha)+p(\alpha)\leq A.$$

By defining a function

$$f(t)=p'(t)(a-t)+(t)$$

for $t\in[\alpha,c]$ and observing that $f(\alpha)\leq A, f(c)\geq A$, Bolzano's theorem 2.3 guarantees an existence of precisely c_a such that $p'(c_a)(a-c_a)+p(c_a)=A$. The existence of c_b can be proven analogously.

To show that $y(x)$ is the shortest path, one can proceed similarly as in the first case of the problem. Assume that the shortest path differs from y at least at one point on $[a,b]$ and construct a new path that is shorter than the path that was assumed to be the shortest one, arriving at a contradiction. Detailed proofs and other cases of the problem can be found in authors paper.[4] □

4 Discussion

This article treats a problem of finding an optimal path uniquely using the methods of mathematical analysis, more precisely, the methods of calculus of variations. Although the article does not provide a definite algorithm for finding the shortest path, it deals with more fundamental and theoretical aspects of the problem such as the length itself and its properties. The very existence of a solution is proven in one of the cases. This method might be extendable to prove the existence of a solution to any case of the problem. Even though the introduced definitions were constructed intuitively, they still had useful mathematical properties. Ideally, in the future research a necessary condition for the shortest path will be discussed and provided. This condition might be in form of a differential equation, similarly to the Euler-Lagrange equation.

5 Conclusion

Basic introduction from an analytical perspective was given along with definitions of basic concepts of the problem, namely the concept of length functional, set of all admissible curves and obstacles. The convexity of the set of all admissible curves and the length functional was demonstrated, which allowed us to prove the existence of a solution in one of the cases. Other case of the problem was considered, where we established under which conditions does a shortest path attain its general form and what this form looks like.

Acknowledgement

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Ready for the future? Then, Teach and Think Critically Teaching Botany Using RWCT Methods

Babyrádová, Veronika[1], Hrežová, Jitka[1]

The main aim of this work was to create a complex educational kit on the topic of poisonous plants growing in the Czech Republic. This topic is often neglected because of the lack of educational materials. The created kit consists of an educational brochure, which serves as a student´s book, complementary worksheets, a board game and a puzzle. Simultaneously, I focused on the creation of an educational kit following an interesting educational method – Reading and Writing for Critical Thinking (RWCT). From my point of view, the partial methods and tasks of critical thinking allow the students to connect the previous knowledge with the newly acquired one and simultaneously consider different perspectives. Hence, the tasks which were created for educational brochures, worksheets and the board game are initiating the interdisciplinary connection between botany and other disciplines, for instance, chemistry, social science or history.

One of the important strengths is removability of the teaching brochure pages, so students can work on one specific poisonous plant, both individually and in groups. Through that, I tried to encourage the development of independence as well as communication and cooperation skills during the group work, all important for their future life. Therefore, it is important to develop these skills intensively during the school years. On the other hand, it is necessary to admit that the RWCT methods are not firmly established in the Czech education system and its use is not a matter of course, although it offers the students a lot of useful things. This fact led me to creating a set which has the potential to guide teachers to use RWCT methods guided by the suggested tasks and the layout, thus simplifying its application during teaching.

The expected qualities of this training set were verified during the detailed planned lesson where the educational set was used along all phases of the three-phase teaching model according to the RWCT methods. I chose to give this lesson to third-year secondary school students (the equivalent of junior year of high school) in their 90-minute biology seminar. First, they performed tasks linking their existing knowledge with the topic of the lesson, then they enriched this knowledge with new contexts, and finally, with the help of a board game, they repeated attractively newly gained information during the memorizing process. In summary, I created an educational set, which, with its tasks and design, encourages teachers to use at least partial elements of the RWCT methods.

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Ready for the future? Then, teach and think critically.

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The main aim of this work was to create a complex educational kit for learning about poisonous plants growing in the Czech Republic. This set is intended especially for juniors at grammar schools, where it may be utilized in biology seminars. The kit aims to showcase interdisciplinary connections between biology and other natural as well as social sciences, which is a way of positively impacting students' interests in the sciences through topics that are not strictly related to biology. Using the RWCT (Reading and Writing for Critical Thinking) method allows the students to connect what they already know with new knowledge, develop a complex opinion thanks to that connection and understand issues that overlap to different fields, which may not be necessarily related. The main contribution of this work lies in the educational set consisting of four parts: the teaching brochure, a board game, a matching game, and several worksheets. The whole set is designed to make the students actively think about the subject, its overlaps, and connections to other fields of study, and then to lead them towards creating a complex opinion on the given topic. Another important feature of the set is that it leads the students to make use of their textbook, rather than any digital sources of information (e.g. slides from a teacher, to which they are accustomed nowadays in classes). The teaching set was tested in a sample biology seminar classes attended by 18 students. Following the lesson, the usefulness of the set was evaluated through semi-structured interviews with three respondents, who had taken part in the class. However, more testing lessons and more similar interviews with attending students would be needed to conclusively prove the effectiveness of this teaching set.

Key words: RWCT methods, interdisciplinary connections, teaching brochure, complexity, universality.

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1 Introduction

The topic of poisonous plants is not only an interesting area of study within the field of biology, but knowledge about it is also an advantage in everyday life. The practical usefulness of the information makes the students think about the topic in a wider context. In secondary education, the students gain this information predominantly from botany posters or from the materials prepared by their teacher (who works with specialist textbooks to make these materials) which are more suited for the students' use. As far as teaching posters go, one example would be a poster published by Scientia Publishing House and translated into Czech by R. Synek, which contains realistic images of chosen poisonous plants that grow in the Czech Republic. The poster gives the name of each pictured plant and also some basic information about poisonous plants at the top part. Another alternative used is a poster published by Klett. When it comes to the sources of specialized scientific information which the students access through their teacher, there are several publications available. For example, a book by J. Novák and K. Hísek called Naše jedovaté rostliny; another called Jedovaté rostliny co-authored by F. Starý and Z. Berger; and a quite recent publication called Plants that Kill: A Natural History of the World's Most Dangerous Plants by E. A. Daucey and S. Larsson. In place of the posters the teachers can also use teaching cards published by Computer Media.[1–3] However, teaching materials in any other language than Czech are hardly ever used in our schools and therefore, a complex teaching material on this topic that would satisfy the needs of biology seminar lessons at grammar schools has so far not been made.

That is why I decided to attempt to create it and incorporate the RWCT methods, which allow the students to view the topic in a broader context and connect newly gained information with what they already know.[4] The main objective of this work was the creation of the teaching set about poisonous plants. It consists of several parts: a teaching brochure with 50 poisonous plants that grow in the Czech Republic and the set is completed with several worksheets, a board game and a matching game.

2 Methodology

2.1 RWCT Methods

"Critical thinking is independent thinking. In a class that is taught via the methods of critical thinking, every person creates their own opinions, values, and convictions. Nobody can do the critical thinking for you."[5]

Critical thinking is a didactic method that not only teaches students to cooperate with one another and in groups, but also to think and formulate ideas independently. These are skills every student should possess. I chose to use the critical thinking methods while creating the botany set because it helps the students precisely develop these skills. Learning to think critically is essential for future generations. At the basis of this approach is the ability to think sceptically about definitive claims and to think about issues and connect them to a variety of other topics. It is also necessary to be able to compare newly acquired knowledge to what one already knows and to consider given topics and issues from various angles, i.e. to include in one's reasoning not only the natural scientific viewpoint but also insights from the humanities. A person who thinks critically can thus think impartially the constituent aspects and different perspectives of a given issue and based on such considerations can evaluate a situation and make an adequate decision.[4]

Based on the study called Critical, Reflective, Creative Thinking and Their Reflections on Academic Achievement published in 2020 it is possible to claim that critical, reflexive and creative thinking have a demonstrably positive influence on each other and an individual's predisposition for academic success.[6]

The ability to think critically can be crucial in gaining one's dream job. In many parts of the world, one of the most important items one would put into a CV are critical thinking and problem-solving skills. An individual's ability to work with others in solving a problem comes from their cognitive and social abilities, i.e. knowledge acquisition and active communication and participation, among others. Apart from the ability to actively cooperate with others, one of the key factors that influence the result of the work being done is the individuality of each participant.[7] The RWCT methods are divided into several phases.

The "Three-phase teaching model" described below respects the mechanisms of natural learning and represents a universal aid in the creation of learning units [8]. Every phase encompasses different RWCT methods, such as brainstorming, freewriting, creation of mind maps, reading with follow-up questions, reading in pairs etc.: 1. Evocation 2. Realization of the meaning of new information 3. Reflection

During the first phase called evocation, the student becomes active, since being active is essential for an effective learning process. The activisation is necessary since students can think independently afterwards while using their own words to express related ideas.[9] In the second phase of the model, the teacher presents the students with new information. At the same time, the students confront the newly gained information that they already know and consider the matter from different points of view. The teacher initiates a discussion in groups, where everyone is actively participating, as well as thinking individually.

In the final phase called reflection, it is crucial that the students form an opinion about the topic because this way they can make sure they have understood it clearly. Based on that the students should be able to speak clearly and fluently about the topic. During group work, connections are created between known and new information and a complex opinion related to the issue is formed.[8] The point of reflection is using the students' experience with the teacher's expertise and maintaining a teacher-student cooperation instead of the standard evaluation of pupils.

2.2 Teaching brochure

One of the most important issues to consider while creating the teaching brochure about poisonous plants using the RWCT methods was the final design and layout of the pages. Firstly, I divided the landscape side layout of the pages into two halves. The first part is informative, and the second part is descriptive. On each page, there is one plant with its name written in Czech and Latin. The plants included in the brochure are divided into groups according to their natural habitat: forests, grasslands, fields and rubble sites, and plants growing in parks and gardens. The important thing is that the pages of the brochure are separate and removable, thus students can work with them in small groups or individually. The brochure includes a list of the 5 groups of toxic substances most common in plants, complete with a short characteristic of each of them. The groups are as follows: glycosides, alkaloids, essential oils and resins, peptides and proteins, and finally tannins. The teaching brochure in Czech is attached to the article in supplementary materials.

2.3 The informative part of the page

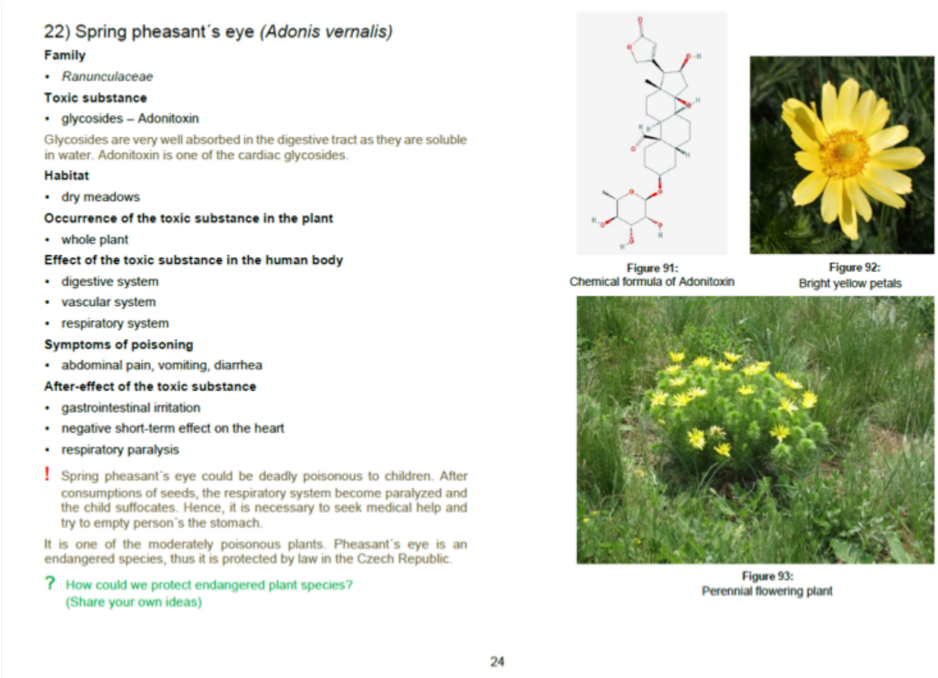
The text located in this part of every page is written both in bullet points and as continuous text. This layout was chosen to prevent the loss of concentration in students. While reading this informative part of the brochure, the reader switches from the bullet-point structure to the continuous text. The text structures are both written in different colours which facilitates students´ concentration. The first continuous text is related to the additional information about the toxic substance contained in a plant. The second continuous text adds interesting information which serves to make the interdisciplinary connections with other fields of study, e.g. with history, like in the case of Datura stramonium. Additional interdisciplinary information makes lessons attractive especially through connections of the new information established between students' field of interest. The text written in bullet points presents basic information about the plant, such as the family, a list of poisons in the plant, the influence of the poison on the human body or symptoms of poisoning.

Clarity was an essential criterion as well, therefore the following two symbols are used in the teaching brochure: a red exclamation mark (!) and a green question mark (?). Generally, a red exclamation mark represents some warning, thus it is used in the brochure with highly poisonous plants or where there is a danger of confusing one plant with another, which is a legitimate concern in real life. Another important aspect was the free space for a discussion and sharing ideas related to the plant initiated by questions or topics which are all marked by the green question mark. Exercises of this type are focused on students' speaking skills because students should be able to speak coherently and present their ideas in a concise manner and to react well when confronted with the questions from the audience. Moreover, the exercise improves the students' ability to formulate appropriate arguments and clarify their opinion on the topic under discussion by analyzing various points of view and considering all pros and cons. At the same time, the exercise teaches the students who are listening to understand the speaker´s opinion. The information and exercises with the green question marks also allow for several options on the part of the teacher. For instance, they can set the students an information-searching exercise as homework and lead a discussion about it in the next lesson, based on their findings. However, it is possible to have the discussion immediately after teaching the material. The green question mark (?) does not always work as a symbol for a discussion. It can also connect the plant with another related kind which differs in the degree of toxicity or it signals an interdisciplinary connection. Hence, I made an effort to create as varied exercises as possible to prevent the students from getting bored.

2.4 Descriptive part of the page

In the descriptive part of the teaching brochure pages, there are chemical structures of compounds present in the plants, pictures of the whole plant and its important morphological characteristics. This part also shows the chemical structures of the most potent poisonous substances found in the plants, which is an example of another interdisciplinary connection as chemistry and biology are closely related. For that reason, it is important to encourage their interdisciplinary connection. For instance, the ability to use knowledge from chemistry to determine parts of a molecule can be crucial for boosting a student's self-confidence. If the student is able to classify the parts of a molecule into main chemical categories based on the typical functional groups, it can positively influence student´s opinion on subjects such as biology and chemistry Another reason why the connection between biology and chemistry is useful is that the combination of chemistry and biology is quite frequent in terms of teacher specializations, and consequently the teacher is capable of answering any follow-up questions the students may have. (Although, this specialization is not a requirement because the teacher can prepare for the questions at home.) Simultaneously, the chemistry-biology combination of seminars is often preferred by grammar school students in their last two years of schooling. It is at these students the educational set is primarily targeted.

Apart from the chemical structures, this part of the page also contains photographs of each of the plants including their botanical description. Photographs are essential for learning or teaching botany. Students are able to connect the information about a plant with its visual representation which is extremely helpful in the process of remembering. The aforementioned landscape layout of the pages and their division into halves was chosen specifically so that the students might see the name and the picture of the plant simultaneously to facilitate the connection of the botanical terminology with the pictures. The photographs capture the natural habitat and the morphologically relevant parts of the angiospermous plants necessary for their correct identification, such as a flower, a leaf or a fruit.



22) Spring pheasant´s eye *(Adonis vernalis)*

Family

- *Ranunculaceae*

Toxic substance

- glycosides – Adonitoxin

Glycosides are very well absorbed in the digestive tract as they are soluble in water. Adonitoxin is one of the cardiac glycosides.

Habitat

- dry meadows

Occurrence of the toxic substance in the plant

- whole plant

Effect of the toxic substance in the human body

- digestive system
- vascular system
- respiratory system

Symptoms of poisoning

- abdominal pain, vomiting, diarrhea

After-effect of the toxic substance

- gastrointestinal irritation
- negative short-term effect on the heart
- respiratory paralysis

! Spring pheasant´s eye could be deadly poisonous to children. After consumptions of seeds, the respiratory system become paralyzed and the child suffocates. Hence, it is necessary to seek medical help and try to empty person´s the stomach.

It is one of the moderately poisonous plants. Pheasant´s eye is an endangered species, thus it is protected by law in the Czech Republic.

? How could we protect endangered plant species? (Share your own ideas)

Figure 91: Chemical formula of Adonitoxin

Figure 92: Bright yellow petals

Figure 93: Perennial flowering plant

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Figure 1: Illustrative one page of the teaching brochure.

2.5 Double worksheets complementing the teaching brochure

Each worksheet, which has the same layout as the brochure pages, contains two plants with the same habitat. I believe that double worksheets will be used primarily for summarizing studied material from the teaching brochure. Worksheets might also be used for introducing a topic and verifying the initial level of the students' knowledge The double worksheets can be viewed in the supplementary materials of the article.

Knowledge testing is not use of these worksheets. The exercises in them are focused on the student's ability to think about a given plant, to connect various pieces of information with what is already familiar to them and to stimulate their interest in the topic. In the exercises, the students are expected to fill in gaps in a text, describe pictures, find and correct mistakes, decide whether statements are true or false and cross-out statements unrelated to the given plant. Such tasks test the knowledge learned during the lesson. The correct answers can be found in the teaching brochure. As I mentioned before, the main aim of the worksheets is to check students´ capabilities and their ability to think in context. This is the reason they also include activities, where students can use their own original ideas, such as mind maps, clover, diamond, or a short written text, all related to one poisonous plant or another. At the same time, these activities might lead the students to formulate questions, which answering of which further deepens their knowledge and understanding of poisonous plants.

2.6 Board game creation

The board game makes the educational set even more attractive. While creating the educational set I was focused on presenting important information effectively as well as attractively. The game is very universal and can be played in different ways and have different purposes, as written in the rules of the game. My main goals while creating the game were that it is informative yet simple and fun to play at the same time. The facts related to 50 poisonous plants are a lot of information to remember and the board game can be an effective tool in the memorization process. For playing the game the factors of cooperation as well as competition are equally important.

2.7 Rules of the game

Students play the game as individuals or in several groups. That way they can practice cooperation and the experience is further intensified by the set time limit. The principle of the game is as follows. A deck of cards with the names of the poisonous plants is put on the table face down. Players roll the dice and move their pieces on the game board from start towards the finish. A player picks a card where the name of a plant is written and fulfils the task which varies depending on the colour of the field where their piece is standing. The solutions to the tasks differ depending on whether the players are individuals or groups.

2.8 The Graphic aspects of the board game

This was another very important aspect to consider because the game should be visually attractive but not distracting, and simple and fun enough to incite students to play. Initially, the intention was to print the game on a folding board, nonetheless, after the first attempt, I realized that if the game must be at least A1 page format, it would be better to print it out on a banner. After the next consideration, I decided to adjust the size of the board to the proportions of the students´ desks. The game area is big and enough to be visible for 20 playing students sitting around the desk.

Figure 2: Visualisation of the board game

2.9 Board game tasks

The game tasks are designed to be entertaining and playful without losing the overall educational purpose. While playing the game, students should be able to communicate with each other using the correct terminology. Playing in groups allows the students to test their new knowledge in a team as well as practice communication and cooperation. The board game can be also used while checking an individual's knowledge. The game tasks include pantomime, drawing morphological characteristics of the plant, enumerating interesting things or describing the effect of a plant's poison on the human body.

2.10 Creation of the Matching Game

The matching game is yet another part of the educational set. It is intended for the higher grades of primary schools, which means pupils at the middle school or junior high school level. The matching game is especially suitable for seventh-grade pupils, who get in touch with botany for the very first time. The topic of poisonous plants and the pupil's ability to recognize them is essential even at this age, and this game allows pupils to naturally connect a picture and name of a poisonous plant since each card contains an illustrative photo of the plant and its name is highlighted in the foreground. The card also contains some additional information, such as what family the given plant belongs to and its Latin name.

The strongest point of the matching game is its variability. The deck of cards includes all 50 poisonous plants mentioned in the teaching brochure but it is possible to choose only those that the teacher deems the most common and the most important plants the students should know, and work only with them during a lesson. This is a good way to teach this topic to younger children. The game is included in the supplementary materials in the version for print as well as previously mentioned partial parts of the educational set.

3 Results and discussion

It was necessary to test the practicality and universality of the created educational set in teaching practice. For this purpose, I prepared a detailed lesson plan for one teaching session where the botany set would be used along with the 3-phase teaching model and the RWCT methods. My observations were focused on whether the students actively participated in the activities, whether they can use accurate terminology and arguments while discussing a given topic, and how successfully they use the teaching brochure.

I chose to give this lesson to third-year secondary school students (the equivalent of junior year of high school) in their 90-minute biology seminar. The seminar session was attended by 18 students from 4 classes of the junior grade of the grammar school in Vyškov. Three of them were subsequently interviewed in relation to their experience of the lesson. These three students are referred to as Respondent A, Respondent B and Respondent C.

In the introductory five minutes, which constituted the evocation phase, the students got acquainted with the topic and drew on their existing knowledge for the first time. The activity employed here was brainstorming. The starting point "poisonous plants" was written on the whiteboard and the students were supposed to immediately add to it whatever they remembered in relation to it. The results then presented the teacher with an estimate of the initial level of knowledge the students have about the topic. The interviews revealed that the students reacted to this activity in different ways. During this task with which they did not have any difficulty, Respondent B noticed a difference in attitudes to it in their classmates: „My classmates were quite hesitant during brainstorming, sort of timid. That is because we are not used to being given space to think and express our thoughts on any topic in a way that does not require clear right or wrong answers (...)." Meanwhile, Respondent C was discomfited by this activity: „(...) during brainstorming we had to go to the board and write things and when it was my turn to go, there had already been a lot written, and I could not think of anything to add." After this, the students were divided into pairs and given the task of formulating questions regarding the topic they would like to have answered. Each pair then chose one question they considered the most relevant. These questions were later answered by the teacher. The teacher as a mentor played a significant role and each of the Respondents noted a marked difference during this seminar session as opposed to a regular class. Respondent A: „We were not force-fed the information, rather, the teacher presented us the individual plants or their effects in an interesting way."

Respondent B emphasized a sense of equality between the teacher and students: „During this lesson, I saw the teacher more as a companion or a partner in learning; it was more about being on the same level with the teacher who was helping us more." Respondent C was aware of a certain level of autonomy during the process: „The teacher seemed to leave us to test ourselves, to play the game, to find out what we knew."." Asking the teacher questions about the topic was also a benefit for the students: „(...) and then I really liked that when we got to ask questions we could think for ourselves about what we were interested in knowing."

The second phase, the realisation of the meaning of the new information, tested what connections the students made in relation to the main topic and how well they could work in pairs. Each pair worked with 3 sheets from the teaching brochure with details about three different poisonous plants. At this point, the method called "insert" was used. This is a method of working with a text, marking the information, and dividing it into 4 groups: already familiar information; new information; known information presented in new ways; things that need to be explained further. They made notes of all this in their notebooks. Respondent B had this to say about such pair work: „You could collaborate with your classmate now, which is not really done in normal lessons (...); I think that if you allow at least 2 people to cooperate that it can give us a lot, we can help one another out." Respondent C agreed with such an assessment: „I definitely like it, interacting with my classmates, because we do not get to work together so much in regular lessons, and I liked working in these groups also because I got paired with a good partner." This respondent's reaction shows how important the pair work can be. On the other hand, Respondent A was really annoyed by having to communicate with others during this stage. What followed was a presentation, where the students shared their findings in pairs and with the rest of the class, and later a discussion. The speaker had to be clear and concise so that his audience would understand their point, which was another way to practice the skill of speaking in front of an audience. The rest of the students passively took in the presented information; however, they could participate actively as well by asking follow-up questions and explaining new terminology. Working with the teaching brochure was a positive experience for respondents A and B, respectively: "I had to look up the information in the brochure and write it down, which helped me to better understand it and remember it."

"What I liked the most was that there was a picture with every new piece of information, which is really helpful for people like me who have more visual memory. I can better learn and remember the relevant information. That happened with several activities during the lesson."

This phase took approximately 30 minutes.

The last phase of the three-phase teaching model, reflection, was carried out via the activity called clover. The students had 5 minutes to showcase what they had learned about the new topic in the following manner: Each pair of students was given a different plant from the set and had the task of defining it with 2 adjectives, then 3 verbs, then one simple 4-word sentence. In the final fifth position, they had to write one word that would best help them identify the plant accurately.

As a final reflective activity, the students played the board game. It took approximately 50 minutes. For the ease of the experience, a simpler variant, where the students had the teaching brochure at hand for looking up information, was chosen. However, they had only 30 seconds to find the answer. First, they were explained the rules of the game and then divided into pairs again, in which they then completed the tasks. The game was quite attractive for Respondent A: "The game was visually interesting for me, (...) it was colourful, and it helped me to navigate it." Respondent C was satisfied with the way the game was played: "The thing I liked the most was the game I think, when we had to look up things and so on."

As an observer in the lesson, I can personally say that the botany lesson was more attractive and even more effective for students than usual, and all three respondents confirmed this as well. The comparison of results of brainstorming and the final reflective clovers also showed progress in the students' knowledge, as they were able to think about what they were learning and became aware of the complexity of the subject matter, which Respondent B also affirmed during the interview: "You learn more that way, remember things better, and you have to think about it in context and in relation to the other topics; and the practical tasks help you to realize you might use this information later in life." Respondent A reacted similarly: "I liked the connection to the other subjects. (...) Here we had Biology connected with Chemistry and it made more sense to me and it was more interesting and more helpful for me."

In the future it would be beneficial to test the versatility and complexity of the educational set in practice on a bigger corpus of sample lessons through interviews. Other adjustments and corrections based on feedback from the students as well as teachers could then result in the gradual incorporation of the set into general teaching practice.

4 Conclusions

The main aim of this work was to create a complex botanical educational set dealing with poisonous plants growing in the Czech Republic following the RWCT methods. An important aspect of this work was also the interdisciplinary connection of biology with other natural sciences as well as humanities, for example with history. The set teaches students to actively connect new information with what they already know and to think in context. Tasks in the teaching brochure, the worksheets, and the boardgame teach the students awareness of such interdisciplinary overlaps and encourage them to form their own comprehensive opinions.

Another reason why this set is an attractive and dynamic teaching aid is that it demonstrates various aspects of the topic of poisonous plants on specific examples. Given the nature of the critical thinking attitudes and methods, the role of the teacher as a mentor is essential, for they must guide the students to the goal of the lesson. Every student begins learning the topic with a different level of existing knowledge and it is crucial that the teachers use this knowledge and teach the students as a group, a team. That way the lesson is productive and the student can get the maximum benefit from it.

The teaching brochure, which is the main output of this work, consists of the following 4 parts: teaching brochure that serves as a textbook, double worksheets, board game and a matching game. One of its main strong points is its universality. The set is primarily made for high school level students, however, parts of it, like the matching game, can be used in lessons at the primary and middle school level. The components of the set can also be successfully combined with all three phases of the critical thinking teaching model-evocation, realization, and reflection. Testing of the set in the teaching practice has proven that it is possible to adjust and work with its respective parts to suit the needs of the students and teachers in order for them to best facilitate the connection between old and new knowledge.

The sample lesson has also corroborated the importance of active student participation and the ability to exchange opinions in a professional objective manner, not only in small work groups but across the class. Discussion led by the teacher prompted the students to create their own comprehensive opinion on the topic, which included biological as well as humanist points of view on poisonous plants. Therefore, it can be stated that the main objective of this work was fulfilled.

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